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Multiple Phases and Return to Equilibrium

A. L. Carey[1] **and L. R. Hume**[1]

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This paper considers the problem of return to equilibrium for perturbations of the dynamics of the one-sided and two-sided XY-models with external field. We find that, in the presence of multiple ground states, return to equilibrium fails for certain perturbations while still holding wen there is a unique ground state.

KEY WORDS: XY-model; return to equilibrium; perturbed dynamics; C^*-algebra; spectral theory.

1. MAIN RESULTS

The problem of return to equilibrium in the XY-model has received considerable attention recently.[1-3,5,6,10] The interesting feature of this model is that it exhibits irreversible behavior which may be analyzed explicitly. Araki and Matsui[4] have also studied the ground states of the XY-chain in an external field and determined the detailed behavior of the set of ground states as a function of the anisotropy and field strength. Of greatest interest is the transition from the regime with two ground states to that with a unique ground state. In comparing the results of ref. 4 with those of ref. 10, it seemed likely that the presence of two ground states rather than one could be detected by irreversible behavior under perturbations of the dynamics. The present study confirms that this is indeed the case.

To describe in more detail how this occurs, consider the XY-Hamiltonian for a one-dimensional spin chain of length $2N+1$ with Hamiltonian

$$H_N = -\frac{1}{4}\left\{\sum_{j=-N}^{N-1} \left[(1+\gamma)\,\sigma_x^{(j)}\sigma_x^{(j+1)} + (1-\gamma)\,\sigma_y^{(j)}\sigma_y^{(j+1)}\right] + 4\lambda \sum_{j=-N}^{N} \sigma_x^{(j)}\right\} \tag{1.1}$$

[1] School of Information Science and Technology, Flinders University of South Australia, and Department of Mathematics, School of MPCE, Macquarie University, Australia.

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Here the $\sigma_\alpha^{(j)}$, where $\alpha = x, y, z$, are the Pauli spin matrices at each site $j \in \mathbb{Z}$. The parameter λ is the field strength and γ is the anisotropy parameter. Both these parameters are real numbers. Let $\mathscr{A}^S$ denote the algebra generated by the Pauli matrices at each site. It is, for the infinite chain, the C^*-algebra consisting of the infinite tensor product of the 2×2 matrices. The time evolution of an observable $A \in \mathscr{A}^S$ defined by[8]

$$\tau_t(A) = \lim_{N \to \infty} e^{itH_N} A e^{-itH_N} \tag{1.2}$$

Our main results concern the return to equilibrium or its failure, as a function of the parameters in the Hamiltonian, after perturbing this dynamics. We consider perturbed one-parameter groups $\{\tau_t^P\}$ where the perturbation is obtained by adding to each H_N a fixed element of $\mathscr{A}^S$ of the type described as local and quadratic in ref. 10; a precise definition is also given below. As an indication of the results, we show, for example, that there exist perturbations P of the dynamics such that return to equilibrium fails to occur for $|\lambda| < 1/2$ but does occur for all other values of λ. In our notation $|\lambda| = 1/2$ is the critical field strength above and at which there is one ground state and below which there are two. Thus, the presence or absence of multiple ground states may be detected by the nonequilibrium behavior of the model. As the behavior of the XY-model near this critical field strength is typical of critical behavior for a wide range of models (the two-dimensional Ising model being one), we anticipate that a similar phenomenon occurs for them as well.

The first step in the analysis of this problem is to introduce the Jordan–Wigner transformation following Araki.[5] First enlarge $\mathscr{A}^S$ by adjoining a new element T having the properties

$$T^2 = 1, \qquad T^* = T, \qquad TAT = \theta_-(A) \quad \text{for } A \in \mathscr{A}^S \tag{1.3}$$

where θ_- is the automorphism of $\mathscr{A}^S$ given by

$$\theta_-(A) = \lim_{N \to \infty} \left(\prod_{j=0}^{-N} \sigma_x^{(j)} \right) A \left(\prod_{j=0}^{-N} \sigma_z^{(j)} \right) \tag{1.4}$$

Within this enlarged algebra, denoted by $\hat{\mathscr{A}}$, we introduce annihilation and creation operators by

$$c_j^* = TS_j(\sigma_x^{(j)} + i\sigma_y^{(j)})/2, \qquad c_j = TS_j(\sigma_x^{(j)} - i\sigma_y^{(j)})/2 \tag{1.5}$$

where

$$\begin{aligned} S_j &= \sigma_z^{(1)} \cdots \sigma_z^{(j-1)} && \text{if } j > 1 \\ S_j &= 1 && \text{if } j = 1 \\ S_j &= \sigma_z^{(0)} \cdots \sigma_z^{(j)} && \text{if } j < 0 \end{aligned}$$

The following relations then hold:

$$[c_j, c_k]_+ = [c_j^*, c_k^*]_+ = 0, \qquad [c_j, c_k^*]_+ = \delta_{jk} 1 \tag{1.6}$$

Thus $\hat{\mathscr{A}}$ also contains the CAR algebra, denoted by $\mathscr{A}^{\text{CAR}}$, generated by these annihilation and creation operators. There is an automorphism of $\hat{\mathscr{A}}$ specified by sending $\sigma_x^{(j)}$ to $-\sigma_x^{(j)}$, $\sigma_y^{(j)}$ to $-\sigma_y^{(j)}$ (and fixing $\sigma_z^{(j)}$). Elements of the algebra are called odd or even, depending on whether they lie in the -1 or $+1$ eigenspace of this automorphism. The intersection of the spin and CAR algebras contains all the even elements of $\mathscr{A}^{\text{CAR}}$ and $\mathscr{A}^{\text{S}}$. In particular, it contains the local Hamiltonians described above, for, as an element of $\hat{\mathscr{A}}$,

$$H_N = \frac{1}{2}\left\{ \sum_{j=-N}^{N-1} \left[(c_j^* c_{j+1} + c_{j+1}^* c_j) + \gamma (c_j^* c_{j+1}^* + c_{j+1} c_j) \right] \right.$$
$$\left. + \lambda \sum_{j=-N}^{N} (2c_j^* c_j - 1) \right\} \tag{1.7}$$

From this it was shown in ref. 5 (also see refs. 8 and 10) that there exists an automorphism group of the enlarged algebra $\hat{\mathscr{A}}$ which extends $\{\tau_t\}$ and furthermore restricts to the CAR algebra as an automorphism group of $\mathscr{A}^{\text{CAR}}$. We will use the same notation for all three. Indeed, the intersection of the CAR and spin algebras remains invariant under the automorphism groups. As in refs. 6 and 10, it follows for each $\beta \in \mathbb{R}$ that there exists a unique (τ, β)-KMS state on $\mathscr{A}^{\text{S}}$ and a unique (τ, β)-KMS state on $\mathscr{A}^{\text{CAR}}$ and that they have the properties that these states agree on the intersection of the two algebras and are identically zero on the odd elements.

Finally, the action of the automorphism group is represented in its simplest form by regarding $\mathscr{A}^{\text{CAR}}$ as a complex Clifford algebra (or self-dual CAR algebra in Araki's[7] terminology). Thus, letting $l_2(\mathbb{Z})$ be the usual sequence space, we introduce $c^*(f) = \sum_{j \in \mathbb{Z}} c_j^* f_j$ and $c(f) = \sum_{j \in \mathbb{Z}} c_j f_j$, where $f = (f_j) \in l_2(\mathbb{Z})$. Then we define $B(h) = c^*(f) + c(g)$, where $h = \binom{f}{g}$. The complex Clifford algebra is generated by $\{B(h)\colon h \in l_2 \oplus l_2(\mathbb{Z})\}$, which satisfy

$$[B(h_1)^*, B(h_2)]_+ = (h_1, h_2) 1, \qquad B(h)^* = B(\Gamma h) \tag{1.8}$$

where

$$\Gamma \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}$$

and $(h_1, h_2) = (f_1, f_2) + (g_1, g_2)$ for

$$h_k = \begin{pmatrix} f_k \\ g_k \end{pmatrix} \quad \text{for} \quad k = 1, 2$$

and $(f_1, f_2) = \sum_{j \in \mathbb{Z}} \bar{f}_{1,j} f_{2,j}$. Now the dynamics is seen to act in a particularly simple way on these operators as

$$\tau_t(B(h)) = B(e^{iKt}h) \tag{1.9}$$

with

$$K = \begin{pmatrix} 2\lambda + (U + U^*)/2 & \gamma(U - U^*)/2 \\ -\gamma(U - U^*)/2 & -2\lambda - (U + U^*)/2 \end{pmatrix} \tag{1.10}$$

where U is the left shift operator on $l_2(\mathbb{Z})$; i.e., $(Uf)_j = f_{j+1}$, $(U^*f)_j = f_{j-1}$.

We now consider perturbations of the XY-dynamics which also act on the CAR-algebra in the manner described for the unperturbed XY-Hamiltonian in (1.12). Such perturbations are the means by which we depart from the given equilibrium state (the unique KMS state for the group τ_t) and which leads us to investigate the question of return to equilibrium.

A first characteristic we require of such a perturbation is that it is implemented by some fixed self-adjoint element P of $\mathscr{A}^{\mathrm{S}}$. The perturbed automorphism group $\{\tau^{\mathrm{P}}\}$ is obtained by adding P to the H_N of (1.1) and taking a limit as in (1.2).[8,10] A second requirement is that the self-adjoint operator is quadratic. This is to mean that $P = P^* \in \mathscr{A}^{\mathrm{S}} \cap \mathscr{A}^{\mathrm{CAR}}$ and $[P, B(h)] = B(Vh)$ for $h \in l_2 \oplus l_2(\mathbb{Z})$, where V is some bounded self-adjoint operator on $l_2 \oplus l_2(\mathbb{Z})$. The first result of these various restrictions is that the automorphism group τ^{P} extends to the larger algebra $\hat{\mathscr{A}}$ and restricts as a $*$-automorphism group of $\mathscr{A}^{\mathrm{CAR}}$. Further, for each $\beta \in \mathbb{R}$ there is a unique $(\tau^{\mathrm{P}}, \beta)$-KMS state on each of the algebras discussed above which agrees on the intersections and is identically zero on the odd elements. Furthermore, the action of the perturbed automorphism group is also simply described by

$$\tau_t^{\mathrm{P}}(B(h)) = B(e^{i(K+V)t}) \tag{1.11}$$

If there is a finite interval $I \subset \mathbb{Z}$ such that $Vh(n) = 0$ for $n \notin I$ and $h \in l_2 \oplus l_2(\mathbb{Z})$, then P is said to be local and quadratic (in the CAR elements) and the corresponding V a local perturbation. This operator V is then finite rank. The smallest interval I in the above is denoted by supp V. The third requirement of the perturbation is that P is local.

In terms of elements of $\mathscr{A}^{\mathrm{S}}$, the local quadratic perturbations include finite linear (but not quadratic) combinations of the $\sigma_z^{(j)}$. Also included are those combinations of Pauli matrices translating to self-adjoint quadratic expressions in the annihilation and creation operators, though the presentation need not be so simple in Pauli operators. Henceforth by a perturbation V we will mean one arising from the requirements of the preceding paragraph on P. Of special interest is the "decoupling" perturbation given by

$$\tfrac{1}{4}[(1+\gamma)\,\sigma_x^0\sigma_x^1 + (1-\gamma)\,\sigma_y^0\sigma_y^1] \tag{1.12}$$

The main result on which all else hinges concerns the spectrum of such perturbations.

Theorem 1.1. Let V denote a self-adjoint local operator. Then the singular continuous spectrum of $K+V$ is empty. The spectrum of $K+V$ consists of a finite number of eigenvalues and, if $\gamma^2 \neq 1$ or $\lambda \neq 0$, one or two closed intervals of $\mathbb{R}$ which are the absolutely continuous spectrum of $K+V$. The absolutely continuous spectrum is empty in the exceptional case.

The general framework for discussing return to equilibrium involves the comparison of the two one-parameter groups of automorphisms of the CAR algebra τ and τ^{P} given by (1.9) and (1.11). The norm limits $\gamma_\pm(A) = \lim_{t\to\pm\infty} \tau^{\mathrm{P}}_{-t}\tau_t(A)$ exist for all elements A of the CAR algebra and are quasifree $*$-morphisms in the sense that there exist bounded operators $O_\pm$ on $l_2 \oplus l_2(\mathbb{Z})$ with $\gamma_\pm(B(h)) = B(O_\pm(h))$, where

$$O_\pm(h) = \lim_{t\to\pm\infty} e^{-it(K+V)}e^{itK}h$$

The range of $O_\pm$ is the subspace corresponding to the absolutely continuous part of the spectrum of $K+V$. Moreover, $\gamma_\pm \tau_t = \tau_t^{\mathrm{P}}\gamma_\pm$ for all $t \in \mathbb{R}$. The inverses $O_\pm^{-1}$ exist on the range of $O_\pm$ and define $*$-morphisms $\gamma_\pm^{-1}$ on the subalgebra of the CAR algebra generated by the $B(h)$ with h in the range of $O_\pm$.

Returning to the Paulion algebra $\mathscr{A}^{\mathrm{S}}$, we again use the same notations τ and τ^{P} to denote the related automorphisms of $\mathscr{A}$; both automorphisms may be regarded as restrictions of the same on a common larger algebra as explained above. Also these groups have a unique KMS state on $\mathscr{A}^{\mathrm{S}}$ for each $\beta \geqslant 0$, and we now denote these by ω_β and $\omega_\beta^{\mathrm{P}}$, respectively. The first fact we deduce is a direct consequence of Theorem 1.1.

Theorem 1.2. Provided $\lambda \neq 0$ or $\gamma^2 \neq 1$, return to equilibrium will occur under the unperturbed dynamics on the two-sided model; that is, for all A in $\mathscr{A}^{\mathrm{S}}$:

$$\lim_{t \to \pm\infty} \omega_\beta^{\mathrm{P}}(\tau_t(A)) = \omega_\beta(A)$$

This is proved in the same way as the corresponding result in ref. 10, which is the result for the case $\lambda = 0$. If we consider evolution under the perturbed dynamics, we may again use identical proofs to those in ref. 10 to obtain the following.

Theorem 1.3. Return to equilibrium will occur under the perturbed dynamics, i.e., for all A in $\mathscr{A}^{\mathrm{S}}$,

$$\lim_{t \to \pm\infty} \omega_\beta(\tau_t^{\mathrm{P}}(A)) = \omega_\beta^{\mathrm{P}}(A)$$

if and only if the spectrum of $K + V$ is purely absolutely continuous. Otherwise we have

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t \omega_\beta(\tau_t^{\mathrm{P}}(B(h)^* B(h'))$$

$$= \omega_\beta^{\mathrm{P}}(B(h)^* B(h')) + \sum_j (\omega_\beta - \omega_P^\beta)(B(P_j h)^* B(P_j h'))$$

where the summation is over the finite number of distinct eigenspaces of $K + V$ with eigenprojections P_j.

Of course the same is true for the one-sided model which is obtained by replacing K by K_+, where K_+ has the same form as K with the one-sided shift on $l^2(\mathbb{Z})$ replacing the two-sided shift everywhere it occurs in (1.11). This leads to our main result.

Theorem 1.4. There exists a finite-rank perturbation P of the XY-Hamiltonian with field for which return to equilibrium occurs under the perturbed dynamics when $|\lambda| \geqslant 1/2$ or $\gamma = 0$ but fails when $|\lambda| < 1/2$ and $\gamma \neq 0$.

Proof. Using Propositions 5.2 and 4.1, we see that $e = 0$ is an eigenvalue of $K + V$ for the local self-adjoint perturbation $V = V_D$ of Eq. (5.7) when $\lambda < 1/2$ and $\gamma \neq 0$. When $\lambda > 1/2$ or $\gamma = 0$ the spectrum of $K_+ + V_D$ is purely absolutely continuous. This is because the spectrum of $K + V_D$ is the same as that of K_+ with the same values of λ and γ. Observe that when P is given by (1.12) one has $[P, B(h)] = B(V_D h)$ for $h \in l_2 \oplus l_2(\mathbb{Z})$. The result then follows from Theorem 1.3. ∎

Remark. In other words, return to equilibrium fails to occur under the perturbed dynamics precisely when there is more than one ground state.

After preliminary facts are established in Section 2, Theorem 1.1 is proved in Sections 3–5 (discrete spectrum) and Section 6 (absence of singular continuous spectrum). As explained above, Theorems 1.2–1.4 then follow as corollaries.

2. PRELIMINARIES

We begin with the spectral analysis of K. Introduce the Fourier transform: $\hat{f}(\theta) = \sum_{j \in \mathbb{Z}} e^{ij\theta} f_j$ for $f = (f_j) \in l_2(\mathbb{Z})$. Then K acts on the Fourier-transformed space by $(\hat{K}h)(\theta) = \hat{K}(\theta)\, \hat{h}(\theta)$, where

$$\hat{K}(\theta) = \begin{pmatrix} 2\lambda + \cos\theta & -i\gamma \sin\theta \\ i\gamma \sin\theta & -2\lambda - \cos\theta \end{pmatrix} \tag{2.1}$$

From this it is clear that if $\lambda \neq 0$ or $\gamma \neq 1$ the spectrum of K is absolutely continuous and is the union of two closed intervals, one in $\mathbb{R}^+ \cup \{0\}$ and one in $\mathbb{R}^- \cup \{0\}$, forming a symmetric subset of $\mathbb{R}$. When $\lambda = 0$ and $\gamma^2 = 1$ the spectrum of K is the set $\{-1, 1\}$ and is thus pure point spectrum.

These intervals depend on the parameters λ, γ. Let

$$I(\lambda, \gamma) = \begin{cases} (|1 - 2|\lambda||, 1 + 2|\lambda||) \\ \qquad \text{if} \quad \gamma^2 = 1 \\ (|1 - 2|\lambda||, |1 + 2|\lambda||) \\ \qquad \text{if} \quad \gamma^2 \neq 1 \text{ and } |2\lambda/(\gamma^2 - 1)| > 1 \\ (\sigma, |1 - 2|\lambda||) \cup (|1 - 2|\lambda||, |1 + 2|\lambda||) \\ \qquad \text{if} \quad \gamma^2 < 1 \text{ and } |2\lambda/(\gamma^2 - 1)| \leqslant 1 \\ (|1 - 2|\lambda||, |1 + 2|\lambda||) \cup (|1 + 2|\lambda||, \sigma) \\ \qquad \text{if} \quad \gamma^2 > 1 \text{ and } |2\lambda/(\gamma^2 - 1)| \leqslant 1 \end{cases} \tag{2.2}$$

and

$$E = \begin{cases} \{z \in \mathbb{C}\colon z^2 = (1 - 2\lambda)^2, (1 + 2\lambda)^2, \sigma\} & \text{when} \quad \gamma^2 \neq 1 \\ \{z \in \mathbb{C}\colon z^2 = (1 - 2\lambda)^2, (1 + 2\lambda)^2\} & \text{when} \quad \gamma^2 = 1 \end{cases} \tag{2.3}$$

where σ denotes the positive square root of $[(4\lambda^2 + \gamma^2 - 1)/(\gamma^2 - 1)]\, \gamma^2$. With this notation the spectrum of K is given by the closure of $-I(\lambda, \gamma) \cup I(\lambda, \gamma)$, unless $\lambda = 0$ and $\gamma^2 = 1$, when it is given by the set $E = \{-1, 1\}$. In all cases some but not necessarily all points of E are included and possibly 0 as well. The points of E are never included in the intervals comprising $-I(\lambda, \gamma) \cup I(\lambda, \gamma)$.

Much of the subsequent analysis of the spectrum of the perturbed Hamiltonian will depend on the properties of polynomial equations arising from $\det(K(z) - e1) = 0$; i.e., the determinant of the matrix in (2.1) after substituting $z = e^{i\theta}$. When $\gamma^2 \neq 1$ one has

$$\det(K(z) - e1) = \frac{\gamma^2 - 1}{4z^2} Q(z)$$

where

$$Q(z) = z^4 - \frac{8\lambda}{\gamma^2 - 1} z^3 + \frac{4e^2 - 16\lambda^2 - 2(1+\gamma^2)}{\gamma^2 - 1} z^2 - \frac{8\lambda}{\gamma^2 - 1} z + 1 \tag{2.4}$$

and when $\gamma^2 = 1$ and $\lambda \neq 0$ one has $\det(K(z) - e1) = (-2\lambda/z)\, Q(z)$, where

$$Q(z) = z^2 + \frac{1 + 4\lambda^2 - e^2}{2\lambda} z + 1 \tag{2.5}$$

It is worth isolating here a number of features of these polynomials for later reference. The roots of the quadratic are given by $r_{1,2} = [l \pm t^{1/2}]/4\lambda$, where $l = e^2 - 1 - 4\lambda^2$ and $t = 1 - 8\lambda^2 + 16\lambda^4 - 2e^2 - 8\lambda^2 e^2 + e^4$. The roots of the quartic are given by

$$r_{1,2} = [2\lambda + m^{1/2} \pm (4\lambda^2 + 4\lambda m^{1/2} + n)^{1/2}]/(\gamma^2 - 1) \tag{2.6}$$

$$r_{3,4} = [2\lambda - m^{1/2} \pm (4\lambda^2 - 4\lambda m^{1/2} + n)^{1/2}]/(\gamma^2 - 1) \tag{2.7}$$

where $m = 4\gamma^2\lambda^2 + \gamma^4 + e^2 - \gamma^2 - \gamma^2 e^2$ and $n = 4\gamma^2\lambda^2 + \gamma^2 + e^2 - 1 - \gamma^2 e^2$. In all cases the roots satisfy the relations $r_1 r_2 = 1 = r_3 r_4$. They are furthermore distinct unless $e \in E$, whereupon repeated roots occur. No root has magnitude one unless e lies in the closure of $-I(\lambda, \gamma) \cup I(\lambda, \gamma)$, the spectrum of K, whereupon at least one has such magnitude. No root is ever zero.

We now study operators of the form $K + V$ on $l_2(\mathbb{Z}) \oplus l_2(\mathbb{Z})$, where V is a self-adjoint local operator. Such operators may have highly nontrivial point spectrum. To analyze this we consider the equation

$$(K + V - e1)\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is equivalent to

$$\det(K(\theta) - e1)\begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} = \begin{pmatrix} e + 2\lambda + \cos\theta & -i\gamma \sin\theta \\ i\gamma \sin\theta & e - 2\lambda - \cos\theta \end{pmatrix} V \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} \tag{2.8}$$

Now let $[N, M]$ be an interval containing supp V. Then we have

$$V\begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{if} \quad n \notin [N, M] \tag{2.9}$$

and hence if $n \notin [N-1, M+1]$

$$\begin{pmatrix} e + 2\lambda + (U + U^*)/2 & \gamma(U - U^*)/2 \\ -\gamma(U - U^*)/2 & e - 2\lambda - (U + U^*)/2 \end{pmatrix} V\begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.10}$$

The implications of these relations can be understood most simply by separating out the case $\gamma^2 = 1$. The general case has the same broad properties as this special case but is considerably more complex. Our strategy of exposition is then to describe the special case in detail and to indicate the changes necessary when this constraint is dropped.

Thus we have for $\gamma^2 = 1$ from (2.10) that the coefficients f_n, g_n satisfy the difference equation, when $n \notin [N-1, M+1]$, given by

$$-2\lambda p(n+1) + (e^2 - 1 - 4\lambda^2)\, p(n) - 2\lambda p(n-1) = 0$$

These equations may be solved in terms of arbitrary initial conditions to give for $m \geqslant 1$

$$f(M+m) = a_1 r^m + a_3 s^m, \qquad g(M+m) = b_1 r^m + b_3 s^m \tag{2.11}$$

and

$$f(N-m) = a'_1 r^m + a'_3 s^m, \qquad g(N-m) = b'_1 r^m + b'_3 s^m \tag{2.12}$$

where r, s are the distinct roots of

$$Q(z) = z^2 + \left(\frac{1 + 4\lambda^2 - e^2}{2\lambda}\right) z + 1 = 0$$

In the case where this polynomial has a repeated root $z = r$, i.e., when $e^2 = (1 + 2\,|\lambda|)^2$ or $e^2 = (1 - 2\,|\lambda|)^2$, then the solutions are

$$f(M+m) = (a_1 + m a_2)\, r^m, \qquad g(M+m) = (b_1 + m b_3)\, r^m \tag{2.13}$$

and

$$f(N-m) = (a'_1 + m a'_3)\, r^m, \qquad g(N-m) = (b'_1 + m b'_3)\, r^m \tag{2.14}$$

The constants a_j, a'_j, b_j, b'_j for $j = 1, 2$ remain to be determined. The condition that the eigenfunction be in the space $l_2(\mathbb{Z}) \oplus l_2(\mathbb{Z})$ naturally constrains these constants and the constraint depends on the values of the roots. We list them in each case:

(a) $r \neq s$, $|r| = 1$. It follows that $|s| = 1/|r| = 1$. Then all constants must be zero to get a normalizable eigenfunction so the eigenfunction has support in $[N-1, M+1]$. Note that in this case the equation $\det(K(\theta) - e1) = 0$ has a solution for θ real, and so e lies in the spectrum of K.

(b) $r \neq s$, $|r| \neq 1$. It follows that neither root has magnitude one and from (1.20) that precisely one has magnitude less than one. Let r denote the smaller root. Then normalizability forces the constants a_3, b_3, a'_3, b'_3 to be zero. Observe further that in this case $\det(K(\theta) - e1)$ has no zeros for θ real; otherwise the polynomial $Q(z)$ would have a root $z = e^{i\theta}$ of modulus one, contrary to our assumption. So e cannot lie in the spectrum of K.

(c) $r = s$. It follows that $r = s = \pm 1$. In this case all constants have to be zero to get a normalizable solution and so the support of the eigenfunction lies in $[N-1, M+1]$ and, as in case 1a, the eigenvalue e must lie in the spectrum of K.

3. THE POINT SPECTRUM—DISSECTION AND THE CASE $\gamma^2 = 1$

The spectral problem of Section 2 may be analyzed further into a problem for a one-sided chain. To see why this is the case, we introduce the projections P_L, P_C, and P_R onto elements of $l_2 \oplus l_2(\mathbb{Z})$ with support in the intervals $(-\infty, N-1]$, $[N, M]$, and $[M+1, \infty)$ respectively. Then the eigenvalue equation

$$(K + V - e1)\begin{pmatrix} f \\ g \end{pmatrix} = 0 \tag{3.1}$$

is equivalent to three other equations. Let

$$\Gamma_+ = \begin{pmatrix} 1/2 & \gamma/2 \\ -\gamma/2 & -1/2 \end{pmatrix}, \qquad \Gamma_- = \begin{pmatrix} 1/2 & -\gamma/2 \\ \gamma/2 & -1/2 \end{pmatrix}$$

The first of these equations is

$$(K - e1)\, P_L \begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} f' \\ g' \end{pmatrix}(n) \tag{3.2}$$

where the vector on the right-hand side has its support limited to $\{N-1, N\}$ with

$$\begin{pmatrix} f'(N-1) \\ g'(N-1) \end{pmatrix} = -\Gamma_+ \begin{pmatrix} f(N) \\ g(N) \end{pmatrix}, \qquad \begin{pmatrix} f'(N) \\ g'(N) \end{pmatrix} = \Gamma_- \begin{pmatrix} f(N-1) \\ g(N-1) \end{pmatrix}$$

The second equation is

$$(K - e1)\, P_R \begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} f'' \\ g'' \end{pmatrix}(n) \tag{3.3}$$

where the vector on the right hand side has its support limited to $\{M, M+1\}$, with

$$\begin{pmatrix} f''(M) \\ g''(M) \end{pmatrix} = \Gamma_+ \begin{pmatrix} f(M+1) \\ g(M+1) \end{pmatrix}, \qquad \begin{pmatrix} f''(M+1) \\ g''(M+1) \end{pmatrix} = -\Gamma_- \begin{pmatrix} f(M) \\ g(M) \end{pmatrix}$$

The third equation is

$$(K + V - e1)\, P_C \begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} f''' \\ g''' \end{pmatrix}(n) \tag{3.4}$$

where the vector on the right-hand side has its support limited to $\{N-1, N, M, M+1\}$ with

$$\begin{pmatrix} f'''(N-1) \\ g'''(N-1) \end{pmatrix} = \Gamma_+ \begin{pmatrix} f(N) \\ g(N) \end{pmatrix}, \qquad \begin{pmatrix} f'''(N) \\ g'''(N) \end{pmatrix} = -\Gamma_- \begin{pmatrix} f(N-1) \\ g(N-1) \end{pmatrix}$$

$$\begin{pmatrix} f'''(M) \\ g'''(M) \end{pmatrix} = -\Gamma_+ \begin{pmatrix} f(M+1) \\ g(M+1) \end{pmatrix}, \qquad \begin{pmatrix} f'''(M+1) \\ g'''(M+1) \end{pmatrix} = \Gamma_- \begin{pmatrix} f(M) \\ g(M) \end{pmatrix}$$

Equations (3.2) and (3.3) may be reformulated as "generalized" eigenvalue problems on the half lattice. To see this, notice that (3.3) implies

$$\begin{aligned} &P_R(K(\lambda, \gamma) - e1)\, P_R \cdot P_R \begin{pmatrix} f \\ g \end{pmatrix} \\ &\quad = \left(-\Gamma_- \begin{pmatrix} f(M) \\ g(M) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right) \end{aligned} \tag{3.5}$$

where $l_2 \oplus l_2(\mathbb{Z}_+)$ is imbedded in the double-sided sequences by identifying $\mathbb{Z}_+$ with $[M+1, \infty)$; $\mathbb{Z}_+$ denotes the nonnegative integers. The operator $P_R K P_R$ restricts to this space and thus can be regarded as acting on the one-sided sequences. In fact $P_R K(\lambda, \gamma)\, P_R = K_+(\lambda, \gamma)$, where K_+ is the same as K except that we replace U everywhere it appears by the one-sided left shift operator U_+ on $l_2(\mathbb{Z}_+)$. That is, $(U_+ f)_n = f_{n+1}$ for $n \geqslant 0$ and the adjoint U_+^* satisfies $(U_+^* f)_0 = 0$ and $(U_+^*) f_n = f_{n-1}$ for $n > 0$, where $f = (f_n) \in l_2(\mathbb{Z}_+)$. A similar set of comments applies to (3.2). If we define S as the reflection map $n \mapsto -n$ and identify $\mathbb{Z}_+$ with $[1-N, \infty)$, then we obtain

$$(K_+(\lambda, -\gamma) - e1)\, SP_L \begin{pmatrix} f \\ g \end{pmatrix} = SP_L(K(\lambda, \gamma) - e1_+)\, P_L S \cdot SP_L \begin{pmatrix} f \\ g \end{pmatrix}$$

$$= \left(-\Gamma_+ \begin{pmatrix} f(N) \\ g(N) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right)$$

Both Eqs. (3.2) and (3.3) thus lead us to solve a problem of the form

$$(K_+(\lambda, \gamma) - e1) \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} x & 0 & 0 & \cdots \\ y & 0 & 0 & \cdots \end{pmatrix} \tag{3.6}$$

Knowledge of the solutions of this problem is employed in Section 5 to detail the point spectrum of the perturbed two-sided operator $K + V$ and later still the one-sided operator as well. Specifically we aim to show the point spectrum is finite. We begin by analyzing Eq. (3.6) and its solutions. The results of this are summarized in a proposition at the end of this section.

Now Eq. (3.6) implies, when $\gamma^2 = 1$, that

$$\begin{aligned} \det(K_+ - e1) \begin{pmatrix} f \\ g \end{pmatrix} &+ \begin{pmatrix} 0 & 2\gamma(1 - U^*U) \\ 2\gamma(1 - U^*U) & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \\ &= \begin{pmatrix} -2\lambda - e - (U_+ + U_+^*)/2 & -\gamma(U_+ - U_+^*)/2 \\ \gamma(U_+ - U_+^*)/2 & 2\lambda - e + (U_+ + U_+^*)/2 \end{pmatrix} \\ &\quad \times \begin{pmatrix} x & 0 & 0 & \cdots \\ y & 0 & 0 & \cdots \end{pmatrix} \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \det(K_+ - e1) = {} & -2\lambda U_+ + (e^2 - 4\lambda^2 - 1)1 - 2\lambda U_+^* \\ & + \tfrac{1}{2}(1 - U_+^* U_+) \end{aligned} \tag{3.8}$$

When $\gamma^2 \neq 1$ the same equation results, with

$$\begin{aligned} \det(K_+ - e1) = {} & \left(\frac{\gamma^2 - 1}{4}\right) U_+^2 - 2\lambda U_+ + \left(\frac{2e^2 - 8\lambda^2 - (1 + \gamma^2)}{2}\right) 1 \\ & - 2\lambda U_+^* + \left(\frac{\gamma^2 - 1}{4}\right) U_+^{*2} + \left(\frac{\gamma^2 + 1}{4}\right) (1 - U_+^* U_+) \end{aligned} \tag{3.9}$$

Observe in (3.7) that the vector on the right is supported on $[0, 1]$ at most. Hence, except possibly when $n \in \{0, 1\}$,

$$\det(K_+ - e1)\, f(n) = 0, \qquad \det(K_+ - e1)\, g(n) = 0 \tag{3.10}$$

As can be seen from Eqs. (3.7)–(3.9), each of the f, g satisfies a difference equation when $\gamma^2 \neq 1$ or $\lambda \neq 0$. This implies each has the form

$$p(n) = \sum_{r_i} \sum_{0}^{o(r_i)-1} a_{ij} n^j r_i^n$$

where the r_i denote the distinct roots of $Q(z)$ given by (1.19) or (1.20), with the choice depending on the values of the parameters λ and γ, and $o(r_i)$ is the multiplicity of the root. The coefficients a_{ij} remain to be determined. Since f, g are square-summable, we obtain some additional restrictions. It will aid the clarity of further exposition to separate out the case $\gamma^2 = 1$ and $\lambda \neq 0$ from the case $\gamma^2 \neq 1$. The latter case has the same broad properties as the former, but involves considerably more tedious calculation. We also deal separately with the case of $\lambda = 0$ and $\gamma^2 = 1$, as it is anomalous.

Thus, when $\gamma^2 = 1$ and $\lambda \neq 0$ we have from (3.7)–(3.10) a difference equation for each set $\{f(n)\}$, $\{g(n)\}$ of the coefficients of the solution for $n > 1$:

$$-2\lambda p(n+1) + (e^2 - 1 - 4\lambda^2)\, p(n) - 2\lambda p(n-1) = 0 \tag{3.11}$$

These equations may be solved in terms of arbitrary initial conditions to give for $m \geqslant 1$

$$f(m) = a_1 r^m + a_2 s^m, \qquad g(m) = b_1 r^m + b_2 s^m \tag{3.12}$$

where r, s are the distinct roots of

$$Q(z) = z^2 + \left(\frac{1 + 4\lambda^2 - e^2}{2\lambda}\right) z + 1$$

In the case where this polynomial has a repeated root $z = r$, i.e., when $e^2 = (1 + 2\,|\lambda|)^2$ or $e^2 = (1 - 2\,|\lambda|)^2$, then the solutions are

$$f(m) = (a_1 + m a_2)\, r^m, \qquad g(m) = (b_1 + m b_2)\, r^m \tag{3.13}$$

These roots have magnitude one only when e lies in the spectrum of K, which in this case is $E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$.

The condition that the solutions be in the space $l_2(\mathbb{Z}_+)$ leads to the following constraints, whose nature depends on the values of the roots:

(a) $e \in -I(\lambda, \gamma) \cup I(\lambda, \gamma)$. Then $r \neq s$, but $|rs| = 1$, and one root has modulus one, hence $|r| = |s| = 1$. Then all constants must be zero to get a normalizable eigenfunction, so the eigenfunction has support only at 0.

(b) $e \notin E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$. Then $r \neq s$, $|r| \neq 1$, $|s| \neq 1$, $|rs| = 1$. It follows that precisely one root, say r, has modulus <1. Then normalizability forces $a_2 = b_2 = 0$.

(c) $e \in E$. It follows that $r = s = \pm 1$. In this case, then, all constants have to be zero to get a normalizable solution and so the support of the eigenfunction is only at 0.

Now we conclude that the solution $\binom{f}{g}$ to (3.6) satisfies:

(a′) If $e \in E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$, the spectrum of K, $\binom{f}{g}(n) = \binom{0}{0}$ for $n \in \mathbb{Z}_+ \backslash \{0\}$.

(b′) If $e \notin E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$, then, for some $a_1, b_1 \in \mathbb{C}$,

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} r^n \qquad \text{for} \quad n \in \mathbb{Z}_+ \backslash \{0\}$$

where r is the unique root of modulus <1 of the polynomial $Q(z)$.

Only case (b′) needs detailed consideration. First some straightforward algebra yields

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = c \begin{pmatrix} \gamma(r + 2\lambda + e) \\ -(r + 2\lambda - e) \end{pmatrix} \tag{3.14}$$

with

$$k = \frac{x - \gamma y}{4\lambda}, \qquad c = \frac{(2\lambda - e)\, y + (2\lambda + e)\, \gamma x}{4\lambda(2\lambda r + 4\lambda^2 - e^2)} \tag{3.15}$$

Hence we conclude that unless both $e = 0$ and $r = -2\lambda$, (3.6) has the unique solution

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} r^n \qquad \text{for} \quad n > 0$$

and (3.16)

$$\begin{pmatrix} f \\ g \end{pmatrix}(0) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \frac{1}{4\lambda} \begin{pmatrix} x - \gamma y \\ \gamma x - y \end{pmatrix}$$

with

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \frac{y(2\lambda - e) + \gamma x(2\lambda + e)}{4\lambda(2\lambda r + 4\lambda^2 - e^2)} \begin{pmatrix} \gamma(r + 2\lambda + e) \\ -(r + 2\lambda - e) \end{pmatrix} \tag{3.17}$$

where r is the unique root of magnitude strictly less than one of the quadratic equation $Q(z)=0$. This solution of (3.6) does indeed lie in $l_2 \oplus l_2(\mathbb{Z}_+)$.

Now returning to our original Eq. (3.6) and substituting as before except with $e=0$ and $r=-2\lambda$, we again find three matrix equations, which lead immediately to the results

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = c\gamma \begin{pmatrix} 1 \\ \gamma \end{pmatrix}, \qquad \begin{pmatrix} f(0) \\ g(0) \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + k \begin{pmatrix} \gamma \\ 1 \end{pmatrix}$$

and

$$2\lambda k\gamma = x, \qquad 2\lambda k = -y$$

where $c, k \in \mathbb{C}$ are some constants. So there is no solution unless $x+y\gamma=0$. If now this extra conditon holds, we find that the solution to (3.6) is given by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = c \begin{pmatrix} \gamma \\ 1 \end{pmatrix} (-2\lambda)^n \qquad \text{for} \quad n>0 \tag{3.18}$$

and

$$\begin{pmatrix} f(0) \\ g(0) \end{pmatrix} = c \begin{pmatrix} \gamma \\ 1 \end{pmatrix} + \frac{1}{2\lambda} \begin{pmatrix} x \\ -y \end{pmatrix} \tag{3.19}$$

It is then easily verified that for any $c \in \mathbb{C}$ this expression is an element of $l_2 \oplus l_2(\mathbb{Z}_+)$.

The case $e=0$ deserves special attention. When $e=0$ we find that the quadratic $Q(z)$ has two roots -2λ, $-1/2\lambda$, distinct when $e \notin E$. Note that r is the unique root of modulus <1. If $0<|\lambda|<1/2$, then $r=-2\lambda$, while if $|\lambda|>1/2$, then $r=-1/2\lambda \neq -2\lambda$. If $|\lambda|=1/2$, then $r=s=\pm 1$ and thus $e \in E$, which is ruled out for the purposes of case (b′). Hence we may conclude that when $\gamma^2=1$, $\lambda \neq 0$, $e \notin E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$, and $e=0$:

(i) If $|\lambda|>1/2$, a unique solution to (3.6) in $l_2 \oplus l_2(\mathbb{Z}_+)$ exists for all $x, y \in \mathbb{C}$. It is given by (3.16)–(3.17).

(ii) If $0<|\lambda|<1/2$ and $x+\gamma y \neq 0$, then no solution to (3.6) exists.

(iii) If $0<|\lambda|<1/2$ and $x+\gamma y=0$, then multiple solutions to (3.6) exist in $l_2 \oplus l_2(\mathbb{Z}_+)$. They are all given by expressions of the form (3.18)–(3.19) with c ranging over $\mathbb{C}$.

(iv) If $0<|\lambda|<1/2$, then K_+ has a one-dimensional eigenspace at $e=0$. The eigenvectors are easily constructed from (3.18)–(3.19).

The analysis is much simpler when e lies in the spectrum of K:

(i′) No solution unless $(2\lambda+e)x+(2\lambda-e)\gamma y=0$.

(ii′) If $(2\lambda+e)x+(2\lambda-e)\gamma y=0$, then a unique solution to (3.6) exists and is given by

$$\begin{pmatrix} f(n) \\ g(n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ if } n>0 \quad \text{and} \quad \begin{pmatrix} f(0) \\ g(0) \end{pmatrix} = \frac{x-\gamma y}{4\lambda} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \tag{3.20}$$

Note that while $e=0$ is allowed in this case it occurs only when $|\lambda|=1/2$. Hence, if $\gamma^2=1$, $\lambda\neq 0$, $e\in E\cup -I(\lambda,\gamma)\cup I(\lambda,\gamma)$, and $e=0$, then we may extend the conclusions above to:

(v) If $|\lambda|=1/2$, no solution to (3.6) exists unless $(2\lambda+e)x+(2\lambda-e)\,y=0$.

(vi) If $|\lambda|=1/2$ and $(2\lambda+e)x+(2\lambda-e)\,y=0$, then a unique solution to (3.6) exists and is given by (3.20).

We still have to deal with Eq. (3.6) in the anomalous case where $\gamma^2=1$ and $\lambda=0$. Straightforward algebra yields in this case that if $e\notin\{0,-1,1\}$, then a unique solution $\binom{f}{g}$ exists for all $x, y\in\mathbb{C}$ and is given by

$$\begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{2(e^2-1)}\left(\begin{pmatrix} h(0) \\ k(0) \end{pmatrix}, \begin{pmatrix} h(1) \\ k(1) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots\right) \tag{3.21}$$

where

$$\begin{aligned} h(0) &= \frac{x+\gamma y-2e^2x}{e}, & k(0) &= \frac{y+\gamma x-2e^2y}{e} \\ h(1) &= \gamma y - x, & k(1) &= y-\gamma x \end{aligned} \tag{3.22}$$

If $e=0$, then no solution exists to (3.6) unless $x+\gamma y=0$ when all solutions $\binom{f}{g}$ satisfy

$$\begin{pmatrix} f \\ g \end{pmatrix} = \left(\begin{pmatrix} c \\ \gamma c \end{pmatrix}, \begin{pmatrix} x \\ \gamma x \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots\right) \tag{3.23}$$

where $c\in\mathbb{C}$. Obviously then $e=0$ is an eigenvalue of K_+ in this case. No solution exists to (3.6) for $e=\pm 1$ unless $x-\gamma y=0$ when all solutions $\binom{f}{g}$ are given by

$$\begin{aligned} \begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{2}\Bigg(&\begin{pmatrix} -2x/e+c_0 \\ -\gamma(2x/e+c_0) \end{pmatrix}, \begin{pmatrix} ec_0+c_1 \\ \gamma(ec_0-c_1) \end{pmatrix}, \ldots, \\ &\begin{pmatrix} ec_n+c_{n+1} \\ \gamma(ec_n-c_{n+1}) \end{pmatrix}, \ldots\Bigg) \end{aligned} \tag{3.24}$$

where $(c_n) \in l_2(\mathbb{Z}_+)$. Hence 1, -1 are eigenvalues e for K_+ when $\gamma^2 = 1$ and $\lambda = 0$.

We now summarize the essential results in the following proposition. Observe that $\sigma(K) = E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$ is the spectrum of the two-sided operator.

Proposition 3.1. When $\gamma^2 = 1$, Eq. (3.6) has the following properties:

(i) e is an eigenvalue iff $e = 0$ and $|\lambda| < 1/2$ or $\lambda = 0$ and $e = \pm 1$.

(ii) For $e \notin (K)$ and not an eigenvalue of K_+ a solution always exists and is unique.

(iii) For $e \in \sigma(K)$ or an eigenvalue of K_+, a solution $\binom{f}{g}$ exists to (3.6) provided x, y satisfy a linear relation. The coefficients of x, y in this relation are functions of λ, γ, e. The solution is then unique unless e is an eigenvalue.

(iv) When e is not an eigenvalue and a solution exists, then the solution $\binom{f}{g}$ satisfies

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = T(n, e, \lambda, \gamma)\begin{pmatrix} x \\ y \end{pmatrix}$$

where T is a rational function of e, r, λ, γ.

(v) The possible eigenspaces at 0, ± 1 are one- and infinite-dimensional, respectively.

4. SPECTRAL ANALYSIS FOR $\gamma^2 \neq 1$

In moving from the Ising model with field ($\gamma^2 = 1$) to the XY-model with field, the complexity of the calculations increases significantly, though the analysis is essentially of the same kind as before. Several new "special" cases will arise, which somewhat obscures the argument. As no further insight is gained from their inclusion, we will omit most of the details and concentrate on the differences from the special case where $\gamma^2 = 1$.

From (3.10) we have that when $n \notin \{0, 1\}$ the coefficients $f(n)$ and $g(n)$ satisfy the difference equation

$$p(n+2) - \frac{8\lambda}{\gamma^2 - 1} p(n+1) + \frac{4e^2 - 16\lambda^2 - 2(1+\gamma^2)}{\gamma^2 - 1} p(n)$$
$$- \frac{8\lambda}{\gamma^2 - 1} p(n-1) + p(n-2) = 0$$

and thus for $n \geqslant 0$ one finds that $f(n)$ and $g(n)$ are given by

$$f(n) = \sum_{r_i} \sum_{0}^{o(r_i)-1} a_{ij} n^j r_i^n, \qquad g(n) = \sum_{r_i} \sum_{0}^{o(r_i)-1} b_{ij} n^j r_i^n \tag{4.1}$$

where r_i denotes the distinct roots of the quartic $Q(z) = 0$ of (1.19), $o(r_i)$ the multiplicity of the root, and a_{ij}, b_{ij} some constants from $\mathbb{C}$. The condition that $f, g \in l_2(\mathbb{Z})$ constrains the coefficients in the expressions (4.1) so that if $|r_i| \geqslant 1$, then $a_{ij} = 0 = b_{ij}$.

We use the notation r, $1/r$, s, $1/s$ to denote the four roots of the quartic with the convention that r and s have the smaller modulus. Note that this notation does not preclude the roots having multiplicity greater than one. With the analysis of the roots of the quartic given in Section 1 in mind, it follows that solutions $\binom{f}{g}$ must take one of the following forms for $n \geqslant 0$:

(A) $\binom{f}{g}(n) = \binom{a_1}{b_1} r^n + \binom{a_2}{b_1} s^n$ if there are two distinct roots of modulus <1.

(B) $\binom{f}{g}(n) = \binom{a_1}{b_1} r^n$ if there is one root of modulus <1 and one of modulus 1.

(C) $\binom{f}{g}(n) = \binom{0}{0}$ when all roots have modulus 1.

(D) $\binom{f}{g}(n) = \binom{a_1}{b_1} r^n + \binom{a_2}{b_2} nr^n$ if there is a repeated root of modulus <1.

The form of possible solutions changes as e varies over $\mathbb{C}$. We detail this:

0(i): $e^2 \in \{(1+2\lambda)^2, (1-2\lambda)^2\}$ and a repeated root, either ± 1, occurs. Forms C or B apply, respectively, when the remaining roots of the quartic have modulus one or not. The circumstances determining which occurs are explained below.

0(ii): $e^2 = [(4\lambda^2 + \gamma^2 - 1)\gamma^2]/(\gamma^2 - 1)$ and all roots are repeated. Forms C or D apply, respectively, when any root has modulus one or none has. This ambivalence is also explained below.

1: $e \notin E \cup \sigma(K) = E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$ and the four roots are distinct and none has modulus one, so that two have modulus less than one. Form A applies in this case.

2: $e \in \sigma(K) \backslash E \cup (-|1+2|\lambda||, -|1-2|\lambda||) \cup (|1-2|\lambda||, |1+2|\lambda||)$ and there are four distinct roots and all have modulus one. Form C applies.

3: $e \in (-|1+2|\lambda||, -|1-2|\lambda||) \cup (|1-2|\lambda||, |1+2|\lambda||)$ and the four roots are distinct, two have modulus one, and precisely one has modulus less than one. Form B applies.

The cases 0(i) and 0(ii) are not necessarily distinct. If the parameters are such that $|2\lambda/(\gamma^2-1)|=1$, then

$$e^2=\frac{4\lambda^2+\gamma^2-1}{\gamma^2-1}\gamma^2\in\{(1+2\lambda)^2, (1-2\lambda)^2\}$$

Otherwise the cases are exclusive, as the points of E are never included in the intervals making up $\pm I(\lambda,\gamma)$. If case 0(ii) occurs and $|2\lambda/(\gamma^2-1)|>1$, then no root has magnitude one, so that there is a single root, of multiplicity two, having magnitude less than one and hence solutions must have form D. If case 0(ii) occurs and $|2\lambda/(\gamma^2-1)|\leqslant 1$, then all roots have magnitude one and hence any solution must have the form C. The case 0(i) is somewhat complicated and we examine the details. In the following circumstances all roots of $Q(z)$ have modulus one and thus solutions to (3.6) must have the form C:

(a) $|2\lambda/(\gamma^2-1)|\leqslant 1$ and $\gamma^2>1$ and $e^2=(1+2\,|\lambda|)^2$.

(b) $|2\lambda/(\gamma^2-1)|\leqslant 1$ and $\gamma^2<1$ and $e^2=(1-2\,|\lambda|)^2$.

These conditions amount to e being an interior point of $\pm I(\lambda,\gamma)$. When $\lambda=0$ or e is in case 0(ii) also—this occurs when $|2\lambda/(\gamma^2-1)|=1$—then all roots again have modulus one and solutions to (3.6) must have the form C. Otherwise exactly one root, not repeated, has modulus less than one and solutions of (3.6) have the form B.

We now move on to calculate the solutions to (3.6). We state the result as follows.

Proposition 4.1.

(a) The generalized eigenvalue problem (3.6) has a one-dimensional affine space of solutions iff $e=0$, $\gamma\neq 0$, and $4\lambda^2<1$ for all x, y satisfying $x+\gamma y=0$.

(b) If $4\lambda^2\geqslant 1$ or $\gamma=0$, there is at most a unique solution $\binom{f}{g}$ to (3.6) when $e=0$. When $0\notin\sigma(K)$ a solution will always exist. When $e=0\in\sigma(K)$ a solution will exist iff x, y satisfy a linear relation determined by λ, γ.

(c) If $e\neq 0$ and $e\notin\sigma(K)$, there is a unique solution $\binom{f}{g}$ to (3.6) for all $x, y\in\mathbb{C}$.

(d) When $e\neq 0$ and $e\in\sigma(K)$ and $Q(z)$ has no roots of modulus one there is no solution unless $x=y=0$, when we have only the zero solution for $\binom{f}{g}$.

(e) When $e\neq 0$ and $e\in\sigma(K)$ and $Q(z)$ has a root of modulus less than one there is a solution to (3.6) provided x, y satisfy certain linear relations. Then there is a unique nonzero solution. The coefficients in these relations are given by polynomial functions of λ, γ, e, and the roots r_i of $Q(z)$ of modulus less than one.

(f) If a solution $\binom{f}{g}$ exists and is unique, then it satisfies matrix equations of the form

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = T(n, e, \lambda, \gamma, r_i)\begin{pmatrix} x \\ y \end{pmatrix}$$

for each $n \in \mathbb{Z}$, with r_i denoting the roots of the polynomial $Q(z)$. The 2×2 matrix T has entries which are rational functions of e, λ, γ, r_i.

(g) The one-sided operator K_+ has an eigenvalue iff $\gamma \neq 0$ and $4\lambda^2 < 1$. This occurs at 0.

Remark. Notice that the first part of the proposition says that e is an eigenvalue of the one-sided operator K_+ only in the region of γ, λ space where there are two ground states.[4]

We sketch the proof as e varies over the regions of $\mathbb{C}$ described above. Some special subcases will arise with particular values of the parameters λ, γ which will complicate the exposition.

Case 1. If a solution exists, it has form A, where, as in Section 3, we may restrict the values of the constants (the a's and b's) by expressing them in terms of x, y, γ, λ, e, and the roots r, s. We introduce the notation

$$\begin{aligned} R_1^\pm &= \frac{r^2}{2} + (2\lambda \pm e)r + \frac{1}{2}, \qquad & S_1^\pm &= \frac{s^2}{2} + (2\lambda \pm e)s + \frac{1}{2} \\ R_2 &= \gamma(r^2 - 1)/2, & S_2 &= \gamma(s^2 - 1)/2 \end{aligned} \tag{4.2}$$

Substituting the form A into (3.6), we obtain for $n = 0$ the equation

$$\begin{pmatrix} \dfrac{R_1^- - 1/2}{r} & \dfrac{S_1^- - 1/2}{s} \\ \dfrac{R_2 + \gamma/2}{r} & \dfrac{S_2 + \gamma/2}{s} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \dfrac{R_2 + \gamma/2}{r} & \dfrac{S_2 + \gamma/2}{s} \\ \dfrac{R_1^+ - 1/2}{r} & \dfrac{S_1^+ - 1/2}{s} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} \tag{4.3}$$

while for $n > 0$ as $r, s \neq 0$ and $r \neq s$ we obtain

$$\begin{pmatrix} R_1^- & R_2 \\ R_2 & R_1^+ \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} S_1^- & S_2 \\ S_2 & S_1^+ \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \tag{4.4}$$

Subcase 1.1. $\gamma = 0$. A brief calculation reveals that the solution $\binom{f}{g}$ to (3.6) must satisfy

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} 0 \\ 2y \end{pmatrix}(t^+)^{n+1} + \begin{pmatrix} -2x \\ 0 \end{pmatrix}(t^-)^{n+1} \qquad \text{for} \quad n \geqslant 0 \tag{4.5}$$

where

$$Q(z) = [z^2/2 + (2\lambda + e)z + 1/2][z^2/2 + (2\lambda - e)z + 1/2]$$

and $t^\pm$ is the root of the first (resp. second) factor of $Q(z)$ with modulus less than one. The vector given by (4.5) is now easily shown to be the unique solution to (3.6). Note that we cannot have $e = 0$ in case 1, for, with $\gamma = 0$, $0 \in E$, which is covered by case 0.

Subcase 1.2. $\gamma \neq 0, \pm 1$. We use (4.4) to eliminate a_1, a_2 from (4.3) to get

$$\begin{pmatrix} \gamma + R_2/R_1^- & \gamma + S_2/S_1^- \\ 1 + \gamma R_2/R_1^- & 1 + \gamma S_2/S_1^- \end{pmatrix}\begin{pmatrix} b_1/2r \\ b_2/2s \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.6}$$

The determinant of this matrix is zero if and only if $R_2 S_1^- - R_1^- S_2 = 0 = S_1^+ R_2 - R_1^+ S_2$, which is equivalent to $e = 0$ and $2\lambda(1 + rs) + r + s = 0$, as $\gamma \neq 0$ and r, s are the two (distinct) roots of modulus less than one. Now we are assuming that $e \notin \sigma(K)$, and hence

$$e^2 \neq (1 \pm 2\lambda)^2 \quad \text{nor} \quad \frac{(4\lambda^2 + \gamma^2 - 1)\,\gamma^2}{\gamma^2 - 1} \tag{4.7}$$

Further notice that $0 \notin -I(\lambda, \gamma) \cup I(\lambda, \gamma)$. Consequently, 0 is within the region covered by case 1 if and only if $1 \pm 2\lambda \neq 0$ and $4\lambda^2 + \gamma^2 - 1 \neq 0$, i.e., if and only if $0 \notin E$. Further, with $e = 0$, the roots of the quartic $Q(z)$ are

$$r_{1,2} = \frac{-2\lambda \pm (4\lambda^2 + \gamma^2 - 1)^{1/2}}{1 + \gamma}, \qquad r_{3,4} = \frac{-2\lambda \pm (4\lambda^2 + \gamma^2 - 1)^{1/2}}{1 - \gamma} \tag{4.8}$$

Note that this labeling above does not agree with that given in Section 1. They are all distinct when $1 \pm 2\lambda \neq 0$ and $4\lambda^2 + \gamma^2 - 1 \neq 0$ and $\gamma^2 \neq 0, 1$. Furthermore, with these conditions applied to λ, γ and with r_i, r_j denoting two of the roots from (4.8), the equation

$$2\lambda(1 + r_i r_j) + r_i + r_j = 0$$

holds if and only if $\{r_i, r_j\} = \{r_1, r_2\}$ or $\{r_3, r_4\}$. It also follows that the determinant of (4.6) in this subcase is zero iff $e = 0$ and $|r_1|$, $|r_2| < 1$ or $e = 0$ and $|r_3|$, $|r_4| < 1$. It remains to be calculated which of the roots in (4.8) have magnitude less than one, that is, whether $\{r, s\} = \{r_1, r_2\}$ or $\{r_3, r_4\}$.

With $e = 0$ and $4\lambda^2 \leqslant 1 - \gamma^2$ then

$$|r_1| = |r_2| = \left|\frac{\gamma - 1}{\gamma + 1}\right|^{1/2} = \frac{1}{|r_3|} = \frac{1}{|r_4|} \neq 1$$

the latter as $\gamma \neq 0$. It is then immediate that $\{r, s\} = \{r_1, r_2\}$ or $\{r_3, r_4\}$—according as γ is greater or less than zero—and hence the determinant is zero. Further, with $e = 0$, it cannot be that $4\lambda^2 = 1 - \gamma^2$ lest e then also lie in the disjoint region covered by case 0(ii). On the other hand, when $e = 0$ and $4\lambda^2 > 1 - \gamma^2$ the constraint $\{r, s\} = \{r_1, r_2\}$ or $\{r_3, r_4\}$ is equivalent to $(r_1^2 - r_4^2)(r_2^2 - r_3^2) > 0$, which simplifies to $1 > 4\lambda^2$.

We may summarize the discussion of this subcase by asserting that the matrix in (4.6) has zero determinant if and only if $e = 0$ and $4\lambda^2 < 1$. Further, in this subcase, it cannot be that $e = 0$ and $4\lambda^2 = 1$ or $e = 0$ and $4\lambda^2 = 1 - \gamma^2$. On the other hand, if $e = 0$ and $1 \leqslant 4\lambda^2$, the determinant is nonzero.

Now (when $e = 0$) the quartic $Q(z)$ factorizes as

$$\frac{1}{1 - \gamma^2}[(1 + \gamma) z^2 + 4\lambda z + (1 - \gamma)][(1 - \gamma) z^2 + 4\lambda z + (1 + \gamma)]$$

which implies $R_1^+ = R_1^- = \pm R_2$ and similarly for the $S_i^\pm$. The roots are so labeled in (4.8) that r_1, r_2 are roots of the first factor and r_3, r_4 of the second. Let $4\lambda^2 < 1$. Since

$$\left|\frac{r_1}{r_3}\right| = \left|\frac{r_2}{r_4}\right| = \left|\frac{1 - \gamma}{1 + \gamma}\right|$$

we have $\{r, s\} = \{r_1, r_2\}$ or $\{r_3, r_4\}$, depending on whether $\gamma > 0$ or $\gamma < 0$. It follows that

$$R_1^+ = R_1^- = -R_2 \quad \text{and} \quad S_1^+ = S_1^- = -S_2 \quad \text{when } \gamma > 0$$
$$R_1^+ = R_1^- = R_2 \quad \text{and} \quad S_1^+ = S_1^- = S_2 \quad \text{when } \gamma < 0$$

Similarly, with $e = 0$ and $4\lambda^2 \geqslant 1$ we have $\{r, s\} = \{r_1, r_3\}$ or $\{r_2, r_4\}$ and hence

$$R_1^+ = R_1^- = -R_2 \quad \text{and} \quad S_1^+ = S_1^- = S_2$$
$$R_1^+ = R_1^- = R_2 \quad \text{and} \quad S_1^+ = S_1^- = -S_2$$

according as r denotes the root of the first factor and s the root of the second factor or conversely. Using these obsrvations, one easily arrives at the following conclusions.

Subcase 1.2.1. $e=0$, $4\lambda^2<1$, $\gamma>0$. There is no solution to (3.6) unless $x+y=0$, whereupon it is given by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n)=\frac{2x}{\gamma-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} r^{n+1}+c\cdot(r^{n+1}-s^{n+1})\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for} \quad n\geqslant 0 \tag{4.9}$$

where r, s are the distinct roots of the factor of the quartic given by $(1+\gamma)\,z^2+4\lambda z+(1-\gamma)$ and c is an arbitrary constant.

Subcase 1.2.2. $e=0$, $4\lambda^2<1$, $\gamma<0$. There is no solution to (3.6) unless $x-y=0$, whereupon it is given by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n)=\frac{2x}{\gamma+1}\begin{pmatrix} -1 \\ 1 \end{pmatrix} r^{n+1}+c\cdot(r^{n+1}-s^{n+1})\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{for} \quad n\geqslant 0 \tag{4.10}$$

where r, s are the distinct roots of the factor $(1-\gamma)\,z^2+4\lambda z+(1+\gamma)$ of $Q(z)$ and c is an arbitrary constant.

Subcase 1.2.3. $e=0$, $|\lambda|>1/2$. The solution to (3.6) is unique and is given by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n)=\frac{y-x}{1-\gamma}\begin{pmatrix} 1 \\ 1 \end{pmatrix} r_+^{n+1}+\frac{y+x}{1+\gamma}\begin{pmatrix} -1 \\ 1 \end{pmatrix} r_-^{n+1} \quad \text{for} \quad n\geqslant 0 \tag{4.11}$$

where r_+ is the (unique) root of modulus less than one of the factor of the quartic given by $(1+\gamma)\,z^2+4\lambda z+(1-\gamma)$ and r_- is the (unique) root of modulus less than one of the factor $(1-\gamma)\,z^2+4\lambda z+(1+\gamma)$ of $Q(z)$.

Subcase 1.2.4. $e\neq 0$. The matrix in Eq. (4.6) cannot have zero determinant if $e\neq 0$. Hence it may be inverted and the solution $\binom{f}{g}$ to (3.6) is seen to be unique and given by

$$\begin{aligned}\begin{pmatrix} f \\ g \end{pmatrix}(n)=&\frac{2r^{n+1}}{G}\begin{pmatrix} -R_1^+S_1^+ & -R_1^+S_2 \\ R_2S_1^+ & S_2R_2 \end{pmatrix}\begin{pmatrix} \gamma & -1 \\ 1 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\\ &+\frac{2s^{n+1}}{G}\begin{pmatrix} S_1^+R_1^+ & S_1^+R_2 \\ -S_2R_1^+ & -R_2S_2 \end{pmatrix}\begin{pmatrix} \gamma & -1 \\ 1 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for} \quad n\geqslant 0\end{aligned} \tag{4.12}$$

where $G=(1-\gamma^2)(R_1^+S_2-S_1^+R_2)$, the entries R_2, S_2, R_1^+, S_1^+ are as in (4.2), and r, s are the two (distinct) roots of the quartic (1.19) whose magnitudes are both less than one.

Case 0(ii). This is the only other case in which the analysis is not straightforward. As before, we devide the discussion into various subcases.

Subcase 0(ii).1. $|2\lambda/(\gamma^2-1)| \leqslant 1$. Here (3.6) has only the trivial solution. Note that if $e^2 = [(4\lambda^2+\gamma^2-1)/(\gamma^2-1)]\,\gamma^2 = 0$ and $\gamma \neq 0$, it cannot be that $|\lambda| < 1/2$ for $4\lambda^2 = 1-\gamma^2$ and hence $|2\lambda/(\gamma^2-1)| \leqslant 1$ implies $|\lambda| \geqslant 1/2$. In fact, it must be that $|\lambda| = 1/2$, since also $4\lambda^2 = 1-\gamma^2 \in [0, 1]$. However, $e=0$, $\gamma=0$, and $|\lambda| < 1/2$ is possible in this subcase.

Subcase 0(ii).2. $|2\lambda/(\gamma^2-1)| > 1$. Form D applies and further subcases are required.

Subcase 0(ii).2.1. $\gamma = 0$. In this subcase $e=0$ and $|\lambda| > 1/2$. Denote the root of modulus <1 by r. Then the solution, by the methods of case 1, is

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \begin{pmatrix} -2x \\ 2y \end{pmatrix} r^{n+1} \qquad \text{for} \quad n \geqslant 0 \tag{4.13}$$

Subcase 0(ii).2.2. $\gamma \neq 0$. Again we must consider subcases.

Subcase 0(ii).2.2.1. $e \neq 0$. The solution $\binom{f}{g}$ to (3.6) is unique and given by, for $n \geqslant 0$,

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \frac{2r^n}{e(1-\gamma^2)}\left[\begin{pmatrix} re & 0 \\ 0 & re \end{pmatrix} + (n+1)\gamma \begin{pmatrix} -R_2 & -R_1^+ \\ R_1^- & R_2 \end{pmatrix}\right] \times \begin{pmatrix} -1 & \gamma \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{4.15}$$

Subcase 0(ii).2.2.2. $e=0$. Observe from the conditions of this subcase—$\gamma^2 \neq 0, 1$ and $0 = e^2 = [(4\lambda^2+\gamma^2-1)/(\gamma^2-1)]\,\gamma^2$—that $4\lambda^2 = 1-\gamma^2$ and thus from the condition $|2\lambda/(\gamma^2-1)| > 1$ we have $0 < |2\lambda| < 1$. Conversely, if $e^2 = [(4\lambda^2+\gamma^2-1)/(\gamma^2-1)]\,\gamma^2 = 0$ and $\gamma^2 \neq 0, 1$ and $|2\lambda| < 1$, we have $|2\lambda/(\gamma^2-1)| > 1$ and $\lambda \neq 0$ for $|2\lambda| > 4\lambda^2 = 1-\gamma^2 > 0$. It follows that when $\gamma > 0$ a solution of (3.6) exists only if $x+y=0$ and is then given for any $d \in \mathbb{C}$ by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \left(\frac{-2\lambda}{1+\gamma}\right)^n \begin{pmatrix} x/\lambda + d(n+1) \\ -y/\lambda + d(n+1) \end{pmatrix} \qquad \text{for} \quad n \geqslant 0 \tag{4.16}$$

When $\gamma < 0$, a solution of (3.6) exists only if $x - y = 0$ and is then given for any $d \in \mathbb{C}$ by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \left(\frac{-2\lambda}{1-\gamma}\right)^n \begin{pmatrix} x/\lambda + d(n+1) \\ -y/\lambda - d(n+1) \end{pmatrix} \qquad \text{for} \quad n \geqslant 0 \tag{4.17}$$

We conclude then that when $e=0$ in this subcase we must have $|\lambda|<1/2$ and the generalized eigenvalue problem has a solution only if the initial conditions satisfy the constraint $x \pm y = 0$, respectively, as γ is greater than or less than zero. When a solution exists there is in fact a one-dimensional affine space of solutions.

Case 2. Here there is no solution unless $x=y=0$ and then it is the trivial solution.

Case 3. If $\gamma \neq 0$, the only solution is

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \frac{2r^{n+1}}{1-\gamma^2}\begin{pmatrix} -1 & \gamma \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{for} \quad n \geqslant 0$$

where r is the unique root of modulus less than one, while if $\gamma=0$, then no solution exists unless either $x=0$ or $y=0$. In either case the unique solution is given by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = 2r^{n+1}\begin{pmatrix} -x \\ y \end{pmatrix} \qquad \text{for} \quad n \geqslant 0$$

Case 0(i). As in the preceding, there are a number of subcases to consider. Observe that in this case if $e=0$, then it must be that $|\lambda| = 1/2$. So we have:

(a) $e^2 = [(4\lambda^2+\gamma^2-1)/(\gamma^2-1)]\,\gamma^2 \quad [=(1 \pm 2\lambda)^2]$ and the only solution to (3.6) is the trivial one.

(b) $\lambda=0$ or equivalently $|1+2\lambda| = |1-2\lambda| = 1$ and only the trivial solution occurs.

(c) $e^2 \neq [(4\lambda^2+\gamma^2-1)/(\gamma^2-1)]\,\gamma^2$ and $\lambda \neq 0$. The solution must have the form C or B according as e lies in the interior of $\pm I(\lambda, \gamma)$ or its boundary. If it has form C, then as above, no solution exists unless $x=y=0$ and it is then the trivial one. When e lies on the boundary and $\gamma=0$, no solution exists unless $x=0$ or $y=0$. When e lies on the boundary and $\gamma \neq 0$, no solution exists unless $(\gamma R_2 + R_1^-)x - (\gamma R_1^- + R_2)\,y = 0$ or equivalently $(R_2 + \gamma R_1^+)x - (\gamma R_2 + R_1^+)\,y = 0$. If either of these sets of conditions is satisfied, the unique solution to (3.6) is given by

$$\begin{pmatrix} f \\ g \end{pmatrix}(n) = \frac{2r^{n+1}}{1-\gamma^2}\begin{pmatrix} -1 & \gamma \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{for} \quad n \geqslant 0$$

This completes the proof of Proposition 4.1, although of course we have proved much more here than is required for that result. ∎

5. SYNTHESIS: EIGENVALUES OF THE TWO-SIDED PROBLEM

In Section 3 we commented that the solution of (3.6) could be used to detail the point spectrum of $K+V$. We begin by reviewing Section 3 in the light of Proposition 4.1. Suppose e is an eigenvalue of $K+V-e1$ with $w=\binom{f}{g}$ a corresponding eigenvector. Then we have that $P_R w$ is a solution in $l_2 \oplus l_2(\mathbb{Z}_+)$ of

$$(K_+(\lambda,\gamma)-e1_+)\begin{pmatrix}h\\k\end{pmatrix}=P_R(K(\lambda,\gamma)-e1)\,P_R\begin{pmatrix}h\\k\end{pmatrix}$$
$$=\left(-\Gamma_-\begin{pmatrix}f(M)\\g(M)\end{pmatrix},0,0,...\right) \tag{5.1}$$

This follows by identifying the space of square-summable sequences on $\mathbb{Z}_+$ with those elements of $l_2 \oplus l_2(\mathbb{Z})$ supported on $[M+1,\infty)$. With S denoting the reflection operator (as in Section 3), one finds K satisfies $SK(\lambda,\gamma)S=K(\lambda,-\gamma)$, so that $SP_L w$ is a solution in $l_2 \oplus l_2(\mathbb{Z}_+)$ of

$$(K_+(\lambda,-\gamma)-e1_+)\begin{pmatrix}h\\k\end{pmatrix}=SP_L(K-e1)\,P_L S\begin{pmatrix}h\\k\end{pmatrix}$$
$$=\left(-\Gamma_+\begin{pmatrix}f(N)\\g(N)\end{pmatrix},0,0,...\right) \tag{5.2}$$

This follows from $SP_L=P_{[-N+1,\infty)}S$ and identifying the space of square-summable sequences on $\mathbb{Z}_+$ with those elements of $l_2 \oplus l_2(\mathbb{Z})$ supported on $[-N+1,\infty)$. Finally, by identifying the range of P_C with $\oplus_N^M \mathbb{C}^2$ in the obvious way, we find that $P_C w$ is a solution in $\oplus_N^M \mathbb{C}^2$ of

$$P_C(K+V-e1)\,P_C\begin{pmatrix}h\\k\end{pmatrix}$$
$$=\left(-\Gamma_-\begin{pmatrix}f(N-1)\\g(N-1)\end{pmatrix},0,0,...,0,0,-\Gamma_+\begin{pmatrix}f(M+1)\\g(M+1)\end{pmatrix}\right) \tag{5.3}$$

This requires an obvious identification of this finite-dimensional space with a subspace of $l_2 \oplus l_2(\mathbb{Z})$. Most importantly, restricted to this subspace $P_C(K+V-e1)\,P_C$ is a matrix operator.

The results of the last section now imply a number of facts about the solutions to these equations. For $e\neq 0, \pm 1$, the one-sided problem has at most one solution for any given initial $\binom{x}{y}$. This is also true for $0, \pm 1$ provided they are not in the point spectrum of $K(\lambda,\gamma)$. Any nonzero eigenvector $w=\binom{f}{g}$ of $K+V-e1$ yields a nonzero vector $P_C w$. Otherwise $f(M)=g(M)=f(N)=g(N)=0$ and thus $SP_L w=P_R w=0$ are the unique solutions of (5.2) and (5.1) with the right-hand side equal to the zero vector. It would then follow w is identically zero. In addition, we have from Proposition 4.1 for e not an eigenvalue of $K_+(\lambda, \pm\gamma)$ that

$$\begin{pmatrix} f(M+1) \\ g(M+1) \end{pmatrix} = -T(0, e, \lambda, \gamma, r_i)\Gamma_- \begin{pmatrix} f(M) \\ g(M) \end{pmatrix} \tag{5.4}$$

$$\begin{pmatrix} f(N-1) \\ g(N-1) \end{pmatrix} = -T(0, e, \lambda, -\gamma, r_i)\Gamma_+ \begin{pmatrix} f(N) \\ g(N) \end{pmatrix} \tag{5.5}$$

where $T(0, e, \lambda, \pm\gamma, r_i)$ is a 2×2 matrix operator with its entries rational functions of e, λ, γ, r_i. The r_i are the roots of the polynomial $Q(z)$ of (2.5)–(2.6).

Finally, $P_C w$ is a nontrivial solution in $l_2 \oplus l_2[N, M]$ of

$$[P_C(K+V-e1)\, P_C - D(e, \lambda, -\gamma)\, P_N - D(e, \lambda, \gamma)\, P_M] \begin{pmatrix} k \\ l \end{pmatrix} = 0 \tag{5.6}$$

where $D(e, \lambda, \gamma) = \Gamma_+ T(0, e, \lambda, \gamma)\, \Gamma_-$, and P_N, P_M denote the projections onto the two-dimensional subspaces consisting of sequences with support in the N and M slots, respectively. Hence the $(M-N+1)\times(M-N+1)$ matrix on the left-hand side of (5.6) has zero determinant.

We have a partial converse to this last result.

Lemma 5.1. Whenever the matrix in (5.6) is singular with $e \notin \sigma(K_+)$, then e must be an eigenvalue of $K+V$.

Proof. Let the matrix in (5.6) be singular. So there is a nontrivial solution $\binom{f_C}{g_C}$ of (5.6) which we identify with its image in $l_2 \oplus l_2(\mathbb{Z})$. With $e \notin \sigma(K_+)$ we may solve the generalized one-sided problem for $K_+(\lambda, \gamma)$ and obtain a unique solution when

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\Gamma_- \begin{pmatrix} f_C(M) \\ g_C(M) \end{pmatrix}$$

Denote the image in $l_2 \oplus l_2[M+1, \infty)$ of this solution by $\binom{f_R}{g_R}$. Similarly we solve the one-sided problem for $K_+(\lambda, -\gamma)$ and get a unique solution when

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\Gamma_+ \begin{pmatrix} f_C(N) \\ g_C(N) \end{pmatrix}$$

Label its image in $l_2 \oplus l_2(-\infty, N-1]$ by $\binom{f_L}{g_L}$. It is easy to see that, with the obvious abuse of notation,

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f_L + f_C + f_R \\ g_L + g_C + g_R \end{pmatrix}$$

satisfies $(K+V-e1)\binom{f}{g} = 0$. Note that $\binom{f}{g}$ is not the zero vector. ∎

The preceding discussion deals with the relationship of the discrete spectra for the one-sided and two-sided problems. There is a straightforward way of relating perturbations of the one-sided problem to perturbations of the two-sided problem which is useful for our main theorem as well as the discussion below. We regard $l_2 \oplus l_2(\mathbb{Z}_+)$ as a subspace of $l_2 \oplus l_2(\mathbb{Z})$, and introduce the shifted reflection map R via

$$R\begin{pmatrix} f(n) \\ g(n) \end{pmatrix} = \begin{pmatrix} f(-1-n) \\ g(-1-n) \end{pmatrix}$$

so that $l_2 \oplus l_2(\mathbb{Z}) = R(l_2 \oplus l_2(\mathbb{Z}_+)) \oplus l_2 \oplus l_2(\mathbb{Z}_+)$. Now define on $l_2 \oplus l_2(\mathbb{Z})$ the operator V_D by

$$\begin{aligned} V_D\begin{pmatrix} f \\ g \end{pmatrix} = \Bigg[& P_{-1}\begin{pmatrix} -U/2 & -\gamma U/2 \\ \gamma U/2 & U^*/2 \end{pmatrix} P_0 \\ & + P_0 \begin{pmatrix} -U^*/2 & \gamma U^*/2 \\ -\gamma U^*/2 & U/2 \end{pmatrix} P_{-1} \Bigg] \begin{pmatrix} f \\ g \end{pmatrix} \end{aligned} \tag{5.7}$$

Then for each perturbation V of K_+ we introduce the perturbation $\tilde{V}$ of K:

$$\tilde{V} = RYVYR + V + V_D \tag{5.8}$$

where Y denotes the operator on $l_2 \oplus l_2(\mathbb{Z})$ defined by the matrix of operators

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proposition 5.2.

(a) $\sigma_{\text{ess}}(K(\lambda, \gamma) + V) = \sigma_{\text{ess}}(K(\lambda, \gamma))$. Thus the spectrum of $K + V$ consists of $\sigma(K)$ together with a discrete set of points in $\mathbb{R}$ which may accumulate at the endpoints of $\sigma(K)$. These points must be eigenvalues of finite multiplicity.

(b) $K(\lambda, \gamma) + \tilde{V} = K_+(\lambda, \gamma) + V + RY[K_+(\lambda, \gamma) + V]\, YR$. Thus the spectrum of $K + \tilde{V}$ is the same as that of $K_+ + V$ and indeed the same is true for each of the absolutely continuous, singular continuous, pure point, essential, and discrete spectra.

(c) $\sigma_{\text{ess}}(K_+ + V) = \sigma_{\text{ess}}(K) = \sigma(K) = \overline{-I(\lambda, \gamma) \cup I(\lambda, \gamma)} = E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$ when $\gamma^2 \neq 1$ or $\lambda \neq 0$. In the case $\gamma^2 = 1$ and $\lambda = 0$ one finds $\sigma_{\text{ess}}(K_+ + V) = \sigma_{\text{ess}}(K + \tilde{V}) = \sigma_{\text{ess}}(K) = E = \{1, -1\}$.

(d) Unless $\gamma^2 = 1$ and $\lambda = 0$ one finds that $\sigma_c(K_+ + V) = \sigma_{\text{ess}}(K_+ + V)$.

(e) When $\gamma^2 = 1$ and $\lambda = 0$ one finds $\sigma_c(K_+ + V)$ and $\sigma_c(K_+)$ are empty.

(f) When $\gamma^2 = 1$ and $\lambda = 0$ one finds $\sigma_c(K + V)$ and $\sigma_c(K)$ are empty.

Corollary 5.3. When $\gamma^2 \neq 1$ or $\lambda \neq 0$ the set $\sigma(K_+)$ equals $E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$ and possibly the point 0. The latter point 0 is included if it is already in E or it is an eigenvalue. Otherwise it is not in the spectrum. It is an eigenvalue iff $\gamma \neq 0$ and $|\lambda| < 1/2$. The former set is the continuous spectrum of K_+.

Corollary 5.4. When $\gamma^2 = 1$ and $\lambda = 0$ the spectrum of $K_+(\lambda, \gamma)$ consists only of eigenvalues with the continuous spectrum being empty. The numbers ± 1 are eigenvalues of infinite multiplicity, and 0 is also an eigenvalue with the corresponding eigenspace being one dimensional. The residual spectrum is also empty.

We omit the proof of this proposition and the corollaries, for in each case the results follow from simple calculations or well-known facts.

Proposition 5.5. For a local perturbation V of K_+, when $\gamma^2 \neq 1$ or $\lambda \neq 0$, one has

$$\sigma_{\text{ac}}(K_+ + V) = \sigma_{\text{ac}}(K + \tilde{V}) = E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$$

For a perturbation V' of K one has

$$\sigma_{\text{ac}}(K + V') = E \cup -I(\lambda, \gamma) \cup I(\lambda, \gamma)$$

Proof. From part 2 of Proposition 3.1 of ref. 10 there is an isometric operator mapping $l_2 \oplus l_2(\mathbb{Z})$ onto the subspace of absolute continuity of the operator $K + V$. This operator Ω_+ satisfies the intertwining properties $\Omega_+ \exp(iKt) = \exp(i[K + V]t)\,\Omega_+$ and $\Omega_+ K = \exp(i[K + V]t)\,\Omega_+$. As the spectrum of K is purely absolutely continuous for any e in the spectrum, there is a sequence of vectors (f_n, g_n) each of norm one which satisfy $\lim_n \|(K - e1)(f_n, g_n)\| = 0$. Then the modified sequence $\langle \Omega_+(f_n, g_n) \rangle$ has its range in the subspace of absolute continuity of $K + V$ and each new vector has norm one also. Furthermore, for any $e \in \sigma(K)$ one now finds

$$\begin{aligned} \|((K + V) - e1)\,\Omega_+(f_n, g_n)\| &= \|\Omega_+(K - e1)(f_n, g_n)\| \\ &= \|(K - e1)(f_n, g_n)\| \xrightarrow{n \to \infty} 0 \end{aligned}$$

It follows from the Weyl criterion that e is in the spectrum of $K + V$ restricted to its subspace of absolute continuity and hence to the absolutely continuous spectrum of $K+V$. The result for the one-sided case follows now from $\sigma_{ac}(K_+ + V) = \sigma_{ac}(K+\tilde{V})$. ∎

Proposition 5.6. Except when $\lambda = 0$ and $\gamma^2 = 1$ all eigenspaces of $K+V$ are finite-dimensional. In the exceptional case the eigenspaces at $e = \pm 1$ are infinite-dimensional but all others have finite dimension. The same statements are true for perturbations of the one-sided operator as well.

Proof. As K has no eigenvalues, it follows that $K - e1$ is one to one on the eigenspace of any eigenvalue e of $K+V$ or indeed any finite-dimensional subspace of the latter. Further, V has a finite-dimensional range. Observe now if $(K+V-e1)(f, g) = (0, 0)$, then $(K-e1)(f, g) = -V(f, g)$ and hence the range of $K-e1$ is contained in the range of V. If the eigenspace were infinite-dimensional, we could choose arbitrarily large subspaces which are mapped one to one by $K-e1$ into this fixed finite-dimensional subspace, a contradiction. ∎

Proposition 5.7. The operator $K+V$ has only finitely many eigenvalues.

The rest of this section is devoted to the proof. We note that this might be expected to follow from general theory as V is only finite rank; however, we have been unable to find such an argument. Consequently we are forced to use a "bare hands" approach. The special case where $\gamma^2 = 1$ is separated from the other cases as it is somewhat simpler computationally and illustrates some of the approach to the more difficult case when $\gamma^2 \neq 1$. The case where $\gamma^2 = 1$ and $\lambda = 0$ is a degenerate case and can be solved by soft methods.

The Case $\gamma^2 = 1$ and $\lambda \neq 0$. We note that K_+ has at most one eigenvalue, at 0, and only when $|\lambda| < 1/2$. Suppose $e \neq 0$ lies in the spectrum of K_+. Then as $\gamma^2 = 1$, we find $D(e, \lambda, \gamma) = 0$ and so if e is an eigenvalue of $K+V$, it is a zero of the polynomial, $\det(P_C K P_C + V - e1) = 0$. There are only finitely many such zeros and so there are only finitely many eigenvalues in the spectrum of $K+V$.

When $e \neq 0$ lies in the complement of the spectrum of K_+ we find that

$$\begin{aligned} D(e, \lambda, \gamma) &= \frac{-r}{4\lambda(r+2\lambda)} \Gamma_+ \\ &\quad \times \begin{pmatrix} 4\lambda(r+2\lambda+e)+er & -\gamma er \\ -\gamma er & er - 4\lambda(r+2\lambda-e) \end{pmatrix} \Gamma_- \\ &= \frac{re}{2\lambda(r+2\lambda)} \begin{pmatrix} -1 & \gamma \\ \gamma & -1 \end{pmatrix} \end{aligned} \tag{5.9}$$

where r is the unique root of

$$2\lambda z^2 + (4\lambda^2 + 1 - e^2)z + 2\lambda = 0 \tag{5.10}$$

with magnitude less than one. As $e \neq 0$, it must be that $r \neq -2\lambda$. The determinant of the matrix in (5.6) then simplifies to

$$\frac{W(e, r, \lambda, \gamma)}{\lambda^4 (r + 2\lambda)^4} \tag{5.11}$$

where W is a polynomial in each of its variables.

It follows from this discussion and the fact that e is an eigenvalue of $K + V$ not equal to 0 and not in $\sigma(K_+)$ that there is a pair (e, z) with z of modulus less than one which simultaneously satisfy (5.10) and $W(e, z, \lambda, \gamma) = 0$. From the algebraic geometry of plane curves there are only finitely many solutions to such equations unless they share a common factor in the ring of polynomials in two variables over $\mathbb{C}$; see, for example, ref. 9. Now observe that if (5.10) factors into a product of polynomials, these polynomials must be either linear or constant in z with coefficients which are polynomials in e independent of z. However, it is easy to see by computing the degrees in e of these coefficients that (5.10) cannot be so factored. Thus the polynomial in (5.10) is irreducible in the ring $\mathbb{C}[e, z]$ and so must divide W. However, if the polynomial in (5.10) divides W, then W will equal zero whenever we find a pair (e, r) satisfying the former equation. As $\lambda \neq 0$, for any $e \notin \sigma(K_+)$ the polynomial equation in z in (5.10) has exactly one root of magnitude less than one and one of magnitude greater than one, neither equal to -2λ. Consequently, for any e we can obtain a pair (e, z) which solves both Eq. (5.10) and $W(e, z, \lambda, \gamma) = 0$ and where $|z| < 1$. It follows that there exists e of arbitrarily large modulus which satisfies (5.6). Clearly then we may choose $e \notin \sigma(K_+)$, so, as claimed above, e must be an eigenvalue of $K + V$. So it follows that we can find eigenvalues of $K + V$ of arbitrarily large modulus, a contradiction. Hence $K + V$ has at most finitely many eigenvalues in $\mathbb{C} \backslash \sigma(K_+)$. We demonstrated earlier there were only a finite number in $\sigma(K_+)$. We make no claim as to whether or not 0 is an eigenvalue.

The Case $\gamma^2 = 1$ and $\lambda = 0$. The result in this case may be proven as in ref. 10.

The Case $\gamma^2 \neq 1$. We partition the complex plane into four sets, some of which may be empty; it will be convenient to define δ as $[\gamma^2(4\lambda^2 + \gamma^2 - 1)/(\gamma^2 - 1)]^{1/2}$. These four sets are as follows:

(i) $E' = E \cup \{0\} = \{0, \pm 1 \pm 2\lambda, \pm \delta\}$.

(ii) We let E_0 be the empty set when $|2\lambda/(\gamma^2-1)| \geqslant 1$. Otherwise, for $|2\lambda/(\gamma^2-1)| \leqslant 1$ and $\gamma^2 \leqslant 1$, E_0 is the set $(-|1+2|\lambda||, -\delta) \cup (\delta, |1+2|\gamma||)$. For $|2\lambda/(\gamma^2-1)| \leqslant 1$ and $\gamma^2 \geqslant 1$, E_0 instead denotes the set $(-\delta, -|1-2|\gamma||) \cup (|1-2|\lambda||, \delta)$.

(iii) $E_1 = (-|1+2|\lambda||, -|1-2|\lambda||) \cup (|1-2|\lambda||, |1+2|\lambda||)$.

(iv) We denote by E_2 the complement of $E' \cup E_0 \cup E_1$.

When $e \in E_0$ we see easily that there are only finitely many solutions. When $e \in E_1$ one finds

$$D(e, \lambda, \gamma) = \frac{r}{2}\begin{pmatrix} -1 & -\gamma \\ \gamma & 1 \end{pmatrix}$$

and hence that the determinant of the matrix in (5.6) yields a polynomial equation $W(e, r, \lambda, \gamma) = 0$. Here r denotes the unique solution z of the quartic equation $Q(e, z, \lambda, \gamma) = 0$ of magnitude less than one. As before, we can conclude from the theory of plain curves that unless these two polynomials in e, z share a common factor, the equations have only finitely many common solutions. Now it is not hard to see that W is a quartic in z and degree $2(M-N+1)$ in e. The coefficient of z^p is a polynomial of degree $2(M-N+1)-p$ or less in e. In fact, the coefficient of z^4 is a polynomial in e of degree exactly $2(M-N-1)$, and that of z^0 is a polynomial in e of degree exactly $2(M-N+1)$. The second polynomial $Q(z)$, a quartic in z, is irreducible in the ring of polynomials in two variables $\mathbb{C}[e, z]$ unless $\gamma^2(4\lambda^2+\gamma^2-1) = 0$. If $\gamma = 0$, this quartic factors as $[z^2+(4\lambda+2e)z+1]$ $[x^2+(4\lambda-2e)z+1]$, while it factors as

$$\left[z^2+\left(\frac{1+e}{\lambda}\right)z+1\right]\left[z^2+\left(\frac{1-e}{\lambda}\right)z+1\right]$$

when $4\lambda^2+\gamma^2-1=0$. If $Q(e, z)$ is irreducible and shares a common factor with $W(e, z)$, then it must be that $W(e, z) = Q(e, z)\, W_2(e, \lambda, \gamma)$, where W_2 is constant with respect to z and a polynomial in e, though not necessarily a polynomial in λ, γ. This means that the coefficient of both z^4 and z^0 in W is the same, i.e., W_2. Yet we have already observed that the coefficients of z^4 and z^0 in W are polynomials in e of differing degrees. So it cannot be in this case that Q is a factor of W in $\mathbb{C}[e, z]$.

Now consider the possibility that λ, γ are such that $Q(e, z)$ factors, and that any of the factors listed above also divides $W(e, z)$ in $\mathbb{C}[e, z]$. The complementary factor, a polynomial in e and z, is a quadratic in z, say $a(e)z^2+b(e)z+c(e)$. Compare $W(e, z)$ calculated as a determinant with a product of this quadratic and any one of the quadratic factors of $Q(e, z)$.

The coefficient of z^0 is $c(e)$ and is a polynomial of degree $2(M-N+1)$, which implies $b(e)$ and then $a(e)$ are polynomials of degree $2(M-N+1)+1$ and $2(M-N+1)+2$, respectively. Yet the coefficient of z^4 is $a(e)$ and must be a polynomial of degree $2(M-N-1)$. So it follows in these circumstances that W and Q cannot share a common factor in $\mathbb{C}[e, z]$.

Hence it follows that Q and W can never have a common factor in $\mathbb{C}[e, z]$ and hence the equations $Q(e, z)=0$ and $W(e, z)=0$ have only finitely many common solutions (e, z). So $H(\lambda, \gamma)+V$ has at most finitely many eigenvaues $e \in E_1$.

Now we come to the case where $e \in E_2$. We find that for $\gamma=0$,

$$T(0, e, \lambda, 0)=\begin{pmatrix} -2t^- & 0 \\ 0 & 2t^+ \end{pmatrix} \tag{5.12}$$

where $t^\pm$ are the unique roots of modulus less than one of $z^2+2(2\lambda \pm e)z+1=0$, respectively. They are also distinct roots of $Q(e, z, \lambda, 0)=0$. When $\gamma \neq 0$,

$$T(0, e, \lambda, \gamma)=\frac{2}{G}\begin{pmatrix} (s-r)\, R_1^+ S_1^+ & sS_1^+ R_2 - rR_1^+ S_2 \\ rR_2 S_1^+ - sS_2 R_1^+ & (r-s)\, R_2 S_2 \end{pmatrix}\begin{pmatrix} \gamma & -1 \\ 1 & -\gamma \end{pmatrix} \tag{5.13}$$

$$=\frac{2}{G'}\begin{pmatrix} (s-r)\, R_2 S_2 & sR_1^- S_2 - rR_2 S_1^- \\ rR_1^- S_2 - sS_1^- R_2 & (r-s)\, R_1^- S_1^- \end{pmatrix}\begin{pmatrix} \gamma & -1 \\ 1 & -\gamma \end{pmatrix} \tag{5.14}$$

where $G=(1-\gamma^2)(R_1^+ S_2 - S_1^+ R_2)$, $G'=(1-\gamma^2)(R_2 S_1^- - S_2 R_1^-)$, and r, s are the distinct roots of $Q(z)$ of modulus less than one and the other notation is as in Section 4.

When $\gamma=0$ and $e \in E_2$ the determinant of the matrix in (5.6) is then seen to expand to a polynomial $W(e, t^+, t^-, \lambda)$ in all the variables. It follows that there is a solution (e, t^+, t^-) of the system of polynomial equations

$$\begin{gathered} W(e, t^+, t^-, \lambda)=0, \qquad (t^+)^2+2(2\lambda+e)\, t^+ + 1 = 0 \\ (t^-)^2 + 2(2\lambda - e)\, t^- + 1 = 0 \end{gathered} \tag{5.15}$$

Conversely, any triple with $e \in E_2$ and $|t^+|$, $|t^-|<1$ will solve the determinant equation when $\gamma=0$. As $e \in E_2$ implies $e \notin \sigma(K_+)$, it follows from Lemma 5.1 that the existence of such a triple (e, t^+, t^-) implies $K+V$ has an eigenvalue e.

When $\gamma \neq 0$ and $e \in E_2$ it can be calculated from (5.13)–(5.14) that the determinant of the matrix in Eq. (5.6) expands as the rational expression $W'(e, r, s, \lambda, \gamma)/G^4=0$, where W' is a polynomial in each of its variables. In fact, when r, s denote the distinct roots of the quartic $Q(e, z, \lambda, \gamma) = 0$, the expression G is always nonzero. It now follows that if the determinant of the matrix in Eq. (5.6) is zero, there is a triple (e, r, s) solving the system of polynomial equations

$$W'(e, r, s, \lambda, \gamma) = 0, \qquad Q(e, r, \lambda, \gamma) = 0, \qquad Q(e, s, \lambda, \gamma) = 0 \tag{5.16}$$

Conversely, any triple with $e \in E_2$, $r \neq s$, $|r|$, $|s| < 1$, will solve the determinant equation when $\gamma \neq 0$, ± 1. As before, the existence of such a triple (e, r, s) implies $K + V$ has an eigenvalue e.

Now, with $\gamma \neq 0$, let S denote the solution set to the system of polynomial equations (5.16) in the variables e, r, s. Suppose $e_0 \in E_2$ is an eigenvalue of $K + V$. Then, as shown earlier, this implies there is a triple $(e_0, r_0, s_0) \in S$ with $r_0 \neq s_0$ and $|r_0|$, $|s_0| < 1$. We consider any such triple in S. As $E_2 = \mathbb{C} \backslash (E' \cup E_0 \cup E_1) = \mathbb{C} \backslash (\sigma(K_+) \cup \{0\})$, S is an open set. It then follows that any $(e_1, r_1, s_1) \in S$ sufficiently near (e_0, r_0, s_0) also satisfies $e_1 \in E_2 \subset \mathbb{C} \backslash \sigma(K_+)$, $r_1 \neq s_1$, and $|r_1|$, $|s_1| < 1$. All of this will imply that e_1, the first coordinate, is an eigenvalue of $K + V$.

Note that E_2 is a subset of the complement of $\sigma_{\text{ess}}(K) = \sigma_{\text{ess}}(K + V)$ and hence any eigenvalue e_0 of $K + V$ is in the latter's discrete spectrum and hence must be an isolated point of the spectrum. It follows that any triple (e_0, r_0, s_0) in S with $e \in E_2$, $r_0 \neq s_0$, and $|r_0|$, $|s_0| < 1$ is an isolated point of S. Using ref. 11, Theorem III.4.4 and Theorem IV.5.1, we see that the set S of solutions is the union of a finite number of irreducible algebraic varieties each of which is a connected subset of $\mathbb{C}^3$ in the Euclidean topology. Hence there can only be a finite number of such triples. Since each eigenvalue e_0 in E_2 gives rise to a distinct triple of this description, it must be that there is only a finite number of eigenvalues of $K + V$ in E_2, when $\gamma \neq 0$. Clearly a virtually identical proof will establish this result when $\gamma = 0$.

With this we have successively established that $K + V$ has a finite number of eigenvalues in each of E_0, E_1, E_2. As E' is a finite set and these four sets partition $\mathbb{C}$, it follows as claimed that $K + V$ has at most a finite number of eigenvalues in $\mathbb{C}$. ∎

6. ABSENCE OF SINGULAR CONTINUOUS SPECTRUM

We have already concluded for the anomalous case of $\gamma^2 = 1$ and $\lambda = 0$ that the continuous and hence absolutely continuous and singular continuous spectra are empty; see Proposition 5.2. In what follows in this section we will assume that either $\gamma^2 \neq 1$ or $\lambda \neq 0$. The remaining cases again involve an elaborate calculation. We begin with some notation. Let $R_V(e)$ and $R(e)$ denote the resolvents of $K + V$ and K, respectively.

Theorem 6.1. Let V be a bounded self-adjoint operator for which

$$\lim_{e \to e_0} \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, R_V(e) \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle$$

exists whenever $\binom{f}{g}$ has finite support, e lies in the upper half-plane, and e_0 is restricted to $-I(\lambda, \gamma) \cup I(\lambda, \gamma) \backslash S'$, where S' is a discrete subset which can only have the points of E as accumulation points. Then the singular continuous spectrum of $K + V$ must be empty.

Proof. We remark first that set containing e_0 is open and in fact is a countable union of open intervals each of which is itself a countable union of cloηintervals. So the set containing e_0 may be written as a countable union of both some closed intervals $[\alpha_i, \beta_i]$ and the corresponding open intervals (α_i, β_i).

We will concentrate our attention on one such interval $[\alpha, \beta]$. By the analyticity of

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, R_V(e) \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle$$

outside the spectrum of $K + V$ the hypothesis that

$$\sup_{0 < \varepsilon < 1} \int_\alpha^\beta \left| \operatorname{Im} \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, R_V(e + i\varepsilon) \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \right|^p de = \infty \tag{6.1}$$

for some $\binom{f}{g}$ of finite support and every $p \in (1, \infty)$ implies that there is some point $e_0 \in [\alpha, \beta]$ and some $\binom{f}{g}$ of finite support for which

$$\lim_{\varepsilon \to 0} \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, R_V(e_0 + i\varepsilon) \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle$$

does not exist. This is contrary to the assumptions of the theorem. So for all $\binom{f}{g}$ of finite support the left-hand side of (6.1) is finite for some $p \in (1, \infty)$. The elements of finite support are dense in $l_2 \oplus l_2(\mathbb{Z})$, so that by ref. 12, Theorem XIII.20, we find $K + V$ has purely absolutely continuous spectrum on the open interval (α, β).

Now as the interval was chosen arbitrarily from the covering of $-I(\lambda, \gamma) \cup I(\lambda, \gamma) \backslash S'$ the singular continuous spectrum must be contained in the union of the discrete set S' with E, the endpoints of the intervals making up $\pm I(\lambda, \gamma)$. The singular continuous spectrum cannot contain isolated points and so the singular continuous spectrum is empty. ∎

Our task is to verify that the hypotheses of the theorem apply to our situation where V is a slf-adjoint local operator. Write the spectral representation of V as

$$V=\sum_{1}^{N} r_j\langle v_j, \cdot\rangle v_j \tag{6.2}$$

for eigenvalues r_j and orthonormal eigenvectors v_j, $j=1, 2,..., N$. These vectors v_j also have finite support within the support of V. Let D be the orthogonal projection onto the range of V.

Lemma 6.2. Whenever $R(e)$ is defined and $1+DR(e)V$ is invertible, $R_V(e)$ also exists and further:

(i) $R_V(e)=R(e)-R(e)\,V(1+DR(e)V]^{-1}\,DR(e)$.

(ii) $DR(e)V=\sum_{i,j=1}^{N} r_j\langle v_i, R(e)\,v_j\rangle\langle v_j, \cdot\rangle\, v_i$.

Proof. Straightforward algebra. ∎

Using (6.2) and Lemma 6.2, it is clear that an inner product

$$\left\langle\begin{pmatrix} f\\ g\end{pmatrix}, R_V(e)\begin{pmatrix} f\\ g\end{pmatrix}\right\rangle$$

may be evaluated as a sum of inner products (i.e., matrix elements) for the operators $R(e)$ and $(1+DR(e)V)^{-1}$. That is, when $\binom{f}{g}$ has finite support, the former inner product may be expanded as a finite sum of terms each a product of matrix elements of these operators between the vector $\binom{f}{g}$ and the vectors of finite support v_j from the representation (6.2) of V. Now if we can show that the matrix elements of $R(e)$ are analytic in e in a neighborhood of $-I(\lambda, \gamma)\cup I(\lambda, \gamma)$, then the analytic Fredholm theorem (ref. 12, Theorem VI.14) implies that the matrix elements

$$\left\langle\begin{pmatrix} f_1\\ g_1\end{pmatrix}, (1+DR(e)V)^{-1}\begin{pmatrix} f_2\\ g_2\end{pmatrix}\right\rangle$$

are meromorphic as functions of e on this same neighborhood. Thus, by Lemma 6.2 the limits required by Theorem 6.1 exist except for a discrete subset of these intervals.

Now observe that by Fourier transforming as in (2.4),

$$\left\langle\begin{pmatrix} f_1\\ g_1\end{pmatrix}, R(e)\begin{pmatrix} f_2\\ g_2\end{pmatrix}\right\rangle$$

may be written, for $\binom{f_j}{g_j}$ of finite support, as a finite sum of integrals of the form

$$\mathrm{Int}(n, e)=\int_0^{2\pi}\frac{e^{in\theta}}{e^2-(\cos\theta+2\lambda)^2-\gamma^2\sin^2\theta}\,d\theta \tag{6.3}$$

where n is an integer and the coefficients are linear functions of the variable e. Moreover, the functions $\mathrm{Int}(n, e)$ are analytic for z in the connected components of the set $R_+ = \{e \mid \Im(e) > 0,\ \Re(e) \in -I(\lambda, \gamma) \cup I(\lambda, \gamma)\}$. Once we establish the next lemma we will have essentially achieved our goal.

Lemma 6.3. All the integrals $\mathrm{Int}(n, e)$ have analytic continuations across the boundary $\pm I(\lambda, \gamma)$ of the components of R_+ to components of $R = \{e \mid \Re(e) \in -I(\lambda, \gamma) \cup I(\lambda, \gamma)\}$.

Proof. These integrals may be evaluated explicitly by splitting the integrand into partial fractions and doing a contour integral in the variable $z = e^{i\theta}$ provided the denominator as a function of z has no roots of modulus one. The latter only occurs when e lies in the closure of $-I(\lambda, \gamma) \cup I(\lambda, \gamma)$. Some, complication could also arise when $e \in E$, i.e., when the roots of the denominator are repeated. However, R_+ does not contain any point of the axes, and hence no point of either of these two sets.

The result of the integral depends on the parameter values and we summarize the computations as follows:

(i) $\gamma^2 \neq 1$. The transformed denominator has four roots when $e \in R_+$, two of modulus greater than one and two less than one. Adopting our previous notation where r, s denote the distinct roots of the quartic $Q(z)$ of modulus less than one, we obtain

$$\mathrm{Int}(n, e) = \frac{8\pi}{1-\gamma^2} \cdot \frac{rs}{(s-r)(1-rs)} \cdot \left[\frac{r^{|n|+1}}{1-r^2} - \frac{s^{|n|+1}}{1-s^2}\right] \tag{6.4}$$

(ii) $\gamma^2 = 1$, $\lambda \neq 0$. The transformed denominator has two (distinct) roots when $e \in R_+$, one of modulus greater than one and one less than one. If r is the root of the quadratic—to which $Q(z)$ reduces—with $|r| < 1$, we obtain

$$\mathrm{Int}(n, e) = \frac{\pi}{\lambda} \cdot \frac{r^{|n|+1}}{1-r^2} \tag{6.5}$$

To facilitate the analytic continuation, we note, by ref. 11, Theorem 3.6, that given a polynomial $p(z, e)$ in two complex variables satisfying $p(z_0, e_0) = 0$ and $(\partial p/\partial z)(z_0, e_0) \neq 0$, there is a neighborhood of (z_0, e_0) within which the set of points (z, e) with $p(z, e) = 0$ forms the graph of a function $z = r(e)$ analytic in a neighborhood of e_0. We apply this in the cases $\gamma^2 \neq 1$ to the quartic $Q(z)$ regarded as a polynomial $Q(z, e)$ in (z, e). We find for any e_0, except where $Q(z)$ has a repeated root as a polynomial in z, which is the set E, that there is a neighborhood U of e_0 and analytic functions r_j, $j=1,\ldots,4$, on U satisfying $Q(r_j(e), e)=0$ for all $e \in U$ with $r_j(e_0)$, $j=1, 2, 3, 4$, being the four distinct roots of $Q(z, e_0)=0$. When $\gamma^2=1$, $\lambda \neq 0$, and hence $Q(z)$ reduces to a quadratic, there are of course only two such functions corresponding to the two distinct roots. Henceforth we deal only with the case $\gamma^2 \neq 1$, the other being similar.

Let S now denote some simply connected, path-connected open set in $\mathbb{C} \backslash E$ and e_0 some point in S and U be some neighborhood of e_0 in S satisfying the above description. Let r_1, r_2, r_3, r_4 denote four such analytic functions on U as described. We may assume the set U is an open ball and as the r_j are distinct at e_0 that for $e \in U$ and $i \neq j$ we in fact have $r_i(e) \neq r_j(e)$ on U. Further note that the points e where $Q(z, e)$ has repeated roots are the elements of the set E and no root equals ± 1 unless it is repeated. We may further assume that the r_i are so numbered that $r_1(e_0)\, r_2(e_0)=1=r_3(e_0)\, r_4(e_0)$ and hence that the other products are not equal to 1. Hence by choosing a smaller ball if necessary we may assume for all $e \in U$ that these other products remain unequal to 1 and hence for all $e \in U$ that $r_1(e)\, r_2(e)=1=r_3(e)\, r_4(e)$.

Given such an open ball U, choose any open set V in S whose intersection with U is connected. Let r be analytic on V with $Q(r(e), e)=0$ for $e \in V$ and let r_j, $j=1,\ldots,4$, be analytic on U and satisfy the properties listed above. Then it follows that for each $e \in U \cap V$, $r(e)=r_j(e)$ for some j which *a priori* may depend on e. As $U \cap V$ is connected, one sees that in fact j is constant over $U \cap V$. This implies for the other possible values of j that $r(e) \neq r_j(e)$ for all e in the intersection.

Now we apply a monodromy-type argument. Let e_0 be some arbitrary point in S and U some open ball in S containing e_0 as described above. Given any path Γ in S with $\Gamma(0)=e_0$, regard it as a complex-valued function on $[0, 1]$; we let r be analytic on U and suppose $Q(r(e), e)=0$ on this ball. Let $s \in [0, 1]$ be the supremum of those t such that (U, r) admits an analytic continuation along $\Gamma[0, t]$ with the analytic continuation satisfying:

(*) There is a finite sequence s_j, $j=0,\ldots, m$, of points in $[0, t]$, with $s_0=0$, $s_m=t$, and open balls $D_j \subseteq S$, $j=0,\ldots, m-1$, respectively, containing $\Gamma([s_j, s_{j+1}])$ and analytic functions w_j defined on each D_j with $w_0(e)=r(e)$ for $e \in D_0 \cap U$ and $w_j(e)=w_{j+1}(e)$ for $e \in D_j \cup D_{j+1} k$.

(**) $Q(w_j(e), e)=0$ for $e \in D_j$.

Now if $s<1$, a straightforward argument along the lines of the previous three paragraphs shows that r may be analytically continued to an open ball D_{m+1} centered on $\Gamma(s)$ and such that both (*) and (**) hold for $j=0,\ldots, m+1$. So we have $s=1$ and that there is a neighborhood of the point $\Gamma(1)$ to which (U, r) may be analytically continued with $(*)$ and $(**)$ holding. By the monodromy theorem (ref. 11, Theorem 8.14) there is a unique function ϕ, analytic on S, extending the function r from U to S. Notice that the map which sends e to $Q(\phi(e), e)$ is analytic in S and zero where ϕ equals r, which includes the open set U and hence is zero everywhere on S.

If we choose the set U sufficiently small so that as above we have four functions r_i analytic on U, distinct at every point of U, and satisfying $r_1(e)\, r_2(e) = 1 = r_3(e)\, r_4(e)$ and $Q(r_i(e), e) = 0$ everywhere on U, we may, as in the preceding paragraph, construct for each i an analytic function ϕ_i on S extending r_i. They will be distinct as ϕ_i extends r_i and all satisfy $Q(\phi_i(e), e) = 0$ everywhere on S. Also, as $\phi_1(e)\, \phi_2(e) - 1$ and $\phi_3(e)\, \phi_4(e) - 1$ are analytic and zero on U, both are also true for all $e \in S$.

The functions ϕ_i so constructed differ at every point of S. For the set of points where any $\phi_i(e) - \phi_j(e) = 0$ is discrete in S or all of S. Suppose there is a point where they are not all different. About such a point we may find a ball $U \subseteq S$ and four functions r'_i on S, distinct and analytic at every point of U and satisfying $Q(r'_i(e), e) = 0$ on U. As $U \cap S = U$ and is convex, a result above shows for each i in turn that there is a unique k such that $\phi_i(e) = r_k(e)$ for all $e \in U$. Since the r_k are everywhere different on U, so are the ϕ_i unless they agree everywhere on U and hence everywhere on S, which is not true. Thus the values of the ϕ_i are distinct at every point and from the last paragraph it must also follow that $\phi_i(e)\, \phi_j(e) = 1$ for any $e \in S$ iff $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$.

Since no root of $Q(z, e)$ is ± 1 unless $e \in E$, it follows that $\phi_j(e) \neq \pm 1$ on such S. It is also true that if S does not meet $-I(\lambda, \gamma) \cup I(\lambda, \gamma) \cup E$, then $Q(z, e)$ has no root of magnitude one and hence for each j either $|\phi_j(e)| > 1$ holds for all $e \in S$ or $|\phi_j(e)| < 1$ holds for all $e \in S$. It will follow in this circumstance that precisely one of ϕ_1, ϕ_2 has magnitude less than one and similarly for the pair ϕ_3, ϕ_4.

By a similar argument we see that the functions so constructed are independent of the original point e_0 and function elements, at least up to reindexing.

We now return to extending the integrals $\mathrm{Int}(n, e)$ of (6.4). Let L_0 denote any one of the open intervals in $-I(\lambda, \gamma) \cup I(\lambda, \gamma)$. Further, let $L = \{z \mid \Re(z) \in L_0\}$ and $L_+ = \{z \mid \Re \in L_0, \Im(z) > 0\}$. The set E is a subset of the axes not meeting $-I(\lambda, \gamma) \cup I(\lambda, \gamma)$ and the latter set does not contain 0. It follows that neither L nor L_+ meets E. By the above arguments we may find four functions ϕ_i on L, analytic and distinct at every point of L, which further satisfy:

(i) $Q(\phi_i(e), e) = 0$ for all i and $e \in L$.

(ii) $\phi_i(e)^2 \neq 1$ for all $e \in L$.

(iii) $\phi_1(e)\,\phi_2(e) = 1 = \phi_3(e)\,\phi_4(e)$ for all $e \in L$ and $\phi_i(e)\,\phi_j(e) \neq 1$ in all other cases.

(iv) $|\phi_1(e)|, |\phi_3(e)| < 1 < |\phi_2(e)|, |\phi_4(e)|$ for all $e \in L_+$.

From this it follows easily that the function $\text{Ext}(n, e)$ defined for $e \in L$ and $n \in \mathbb{Z}$ by

$$\text{Ext}(n, e) = \frac{8\pi}{1-\gamma^2} \frac{\phi_1(e)\,\phi_3(e)}{[\phi_1(e) - \phi_3(e)][1 - \phi_1(e)\,\phi_3(e)]} \times \left[\frac{\phi_1^{|n|+1}}{1-\phi_1(e)^2} - \frac{\phi_3^{|n|+1}}{1-\phi_3(e)^2}\right]$$

equals $\text{Int}(n, e)$ for $e \in L_+$ and is furthermore well defined and analytic everywhere on L.

This completes the proof of Lemma 6.3 in the case $\gamma^2 \neq 1$. As indicated earlier, the method of proof of the result when $\gamma^2 = 1$ and $\lambda \neq 0$ is the same. ∎

Finally we need to verify the hypotheses of Theorem 6.1. Let L now denote the set $\{z\colon \Re(z) \in -I(\lambda, \gamma) \cup I(\lambda, \gamma)\}$ and L_+ now be the set $\{z\colon \Im(z) > 0, \Re(z) \in -I(\lambda, \gamma) \cup I(\lambda, \gamma)\}$. By Lemma 6.3 we now know that

$$\lim_{\varepsilon \to 0} \left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, R(e_0 + i\varepsilon) \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle$$

exists for $e_0 \in -I(\lambda, \gamma) \cup I(\lambda, \gamma)$, $\varepsilon > 0$, and functions $\binom{f_j}{g_j}$ of finite support and that as a function of e,

$$\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, R(e) \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle$$

admits an analytic extension from L_+ to the set L. Combined with Lemma 6.2 and the representation of V in Eq. (6.2), we then find that $DR(e)V$ admits an analytic extension from R_+ to R, as a (finite-rank) operator-valued function. As $|e| \to \infty$, $R(e)$ converges to zero in norm and hence $DR(e)V$ does also, which implies that $(1 + DR(e)V)^{-1}$ exists for all e with norm sufficiently large. If $Z(e)$ denotes the analytic extension of $DR(e)V$ from L_+ to L as a finite-rank operator, then we conclude that $(1 + Z(e))^{-1}$ exists for some e in each connected component of L. The analytic Fredholm theorem then implies that as a function of e,

$(1+DR(e)V)^{-1}$ and hence exists for all $e \in L$ except for a discrete subset. It follows that except for a discrete set of points $e_0 \in -I(\lambda, \gamma) \cup I(\lambda, \gamma)$,

$$\lim_{\varepsilon \to 0} \left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, (1+DR(e)V)^{-1} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle$$

exists for $\varepsilon > 0$. By Lemma 6.2 the existence of these limits above in the described circumstances shows that the conditions of Theorem 6.1 may be satisfied and thus the singular continuous spectrum is empty whenever $\gamma^2 \neq 1$ or $\lambda \neq 0$.

As indicated at the start of this section, the continuous and hence singular continuous part of the spectrum of $K+$ is empty when $\gamma^2 = 1$ and $\lambda = 0$. Hence we have now proved Theorem 1.1. The same result also holds for self-adjoint local perturbations of K_+. Hence Proposition 5.7 now implies that Theorem 1.1 also holds for $K_+ + V$.

Finally we note the following corollary of our arguments.

Corollary 6.4. The spectrum of $K_+(\lambda, \gamma)$ contains an eigenvalue at 0 iff $\gamma \neq 0$ and $|\lambda| < 1/2$. Otherwise the spectrum contains no other eigenvalues unless $\gamma^2 = 1$ and $\lambda = 0$, when there are two more eigenvalues at ± 1. The singular continuous spectrum is always empty, as is the absolutely continuous spectrum when $\gamma^2 = 1$ and $\lambda = 0$. In the other cases the absolutely continuous spectrum consists of one or two closed intervals of $\mathbb{R}$.

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